Chapter 4
Time Response

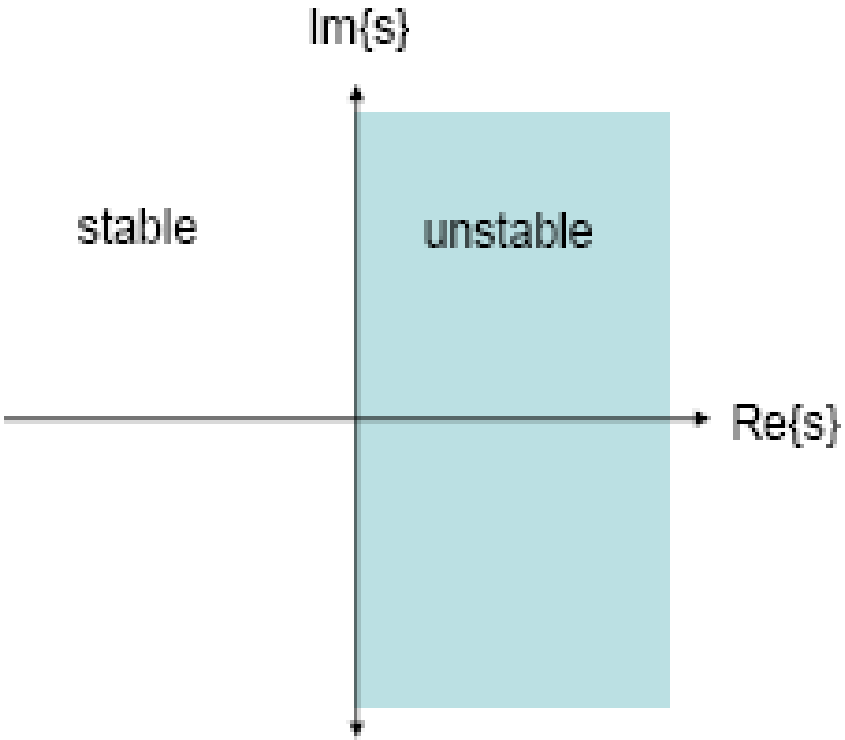
Goals

- Find the time response from the transfer function.
- Use poles and zeros to determine the response.
- Describe quantitatively the transient response of 1^{st} or 2^{nd} order systems.
- Approximate higher-order systems as 1^{st} and 2^{nd} order.
- View the effects of non-linearities on the system time response.
- Find the time response from the state-space representation.

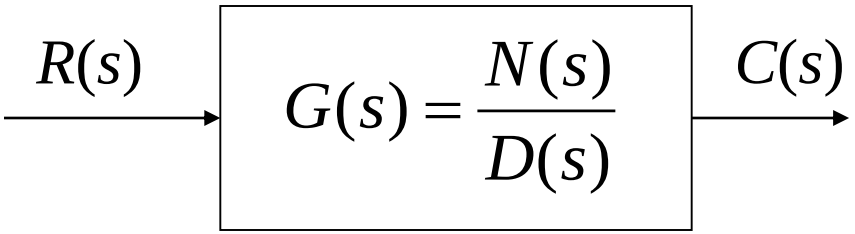
Chapter 4
Time Response

A qualitative understanding of the effect of poles and zeros on the system response can help us to quickly estimate (Schätzung)

Performance (Geschehen)



4.2 Poles, Zeros, and System Response



- Poles: Roots of the denominator.

  Values of s that make $G(s)$ become infinite.
- Zeros: Roots of the numerator.

  Values of s that make $G(s)$ become zero.

4.2 Poles, Zeros, and System Response

Consider the step response of a first-order system.

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1} \qquad let \quad R(s) = \frac{1}{s} \ \ (unit \ \ step)$$

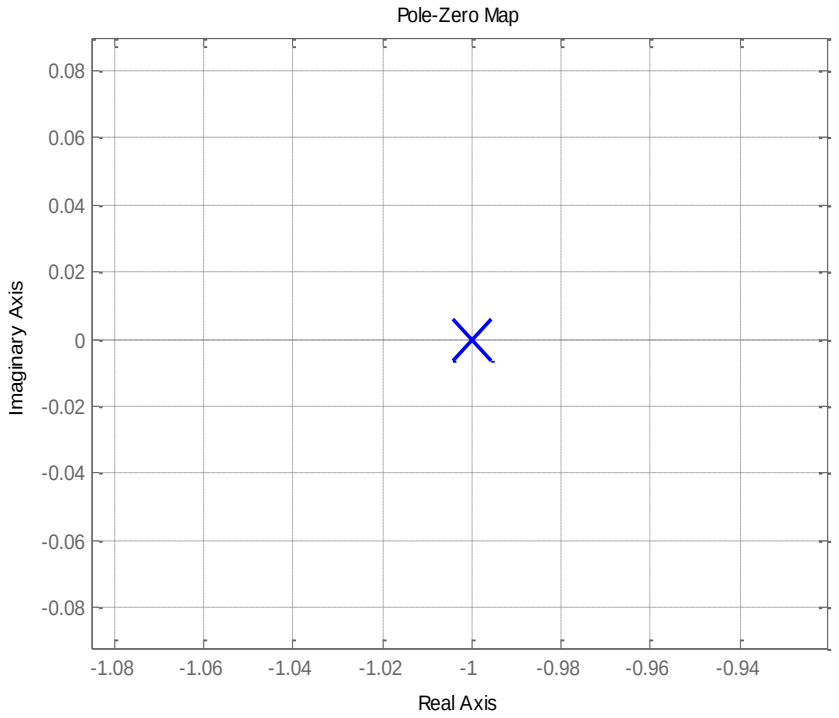
$$\Rightarrow C(s) = \frac{1}{s}\frac{1}{Ts+1} \Rightarrow C(s) = \frac{1}{s} - \frac{1}{Ts+1}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+1/T}$$

$$\Rightarrow c(t) = 1 - e^{-t/T}$$

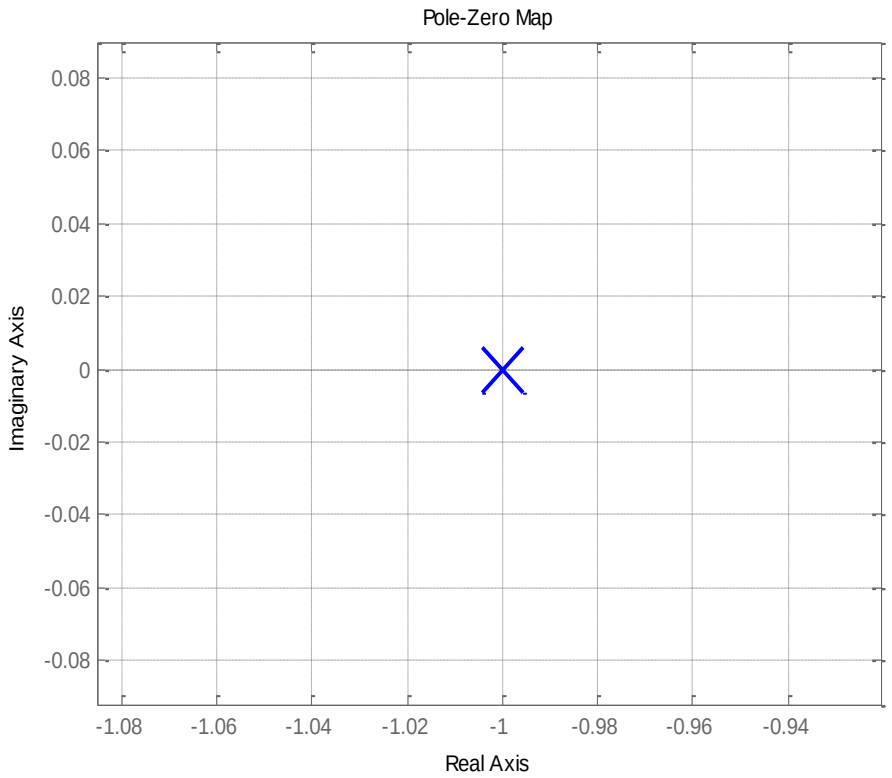
Force response

Natural response



4.2 Poles, Zeros, and System Response

$$c(t) = 1 - e^{-t/T}$$



A pole of the input generates(تَسَبَّبَ) the form of the forced response.

A pole of the transfer function generates the form of the natural response.

All poles and zeros contribute(ساهم) to the amplitudes of both responses.

- The farther to the left a poles on the negative real axis, the faster the exponential transient response will decay to zero.

Time Response

- A first order system without zeros can be described by:

$$G(s) = \frac{1}{Ts + 1}$$

- The resulting impulse response is $g(t) = e^{-t/T}$
- When
  - 1/T > 0, pole is located at s < 0, exponential decays = stable
  - 1/T < 0, pole is at s > 0, exponential grows = unstable

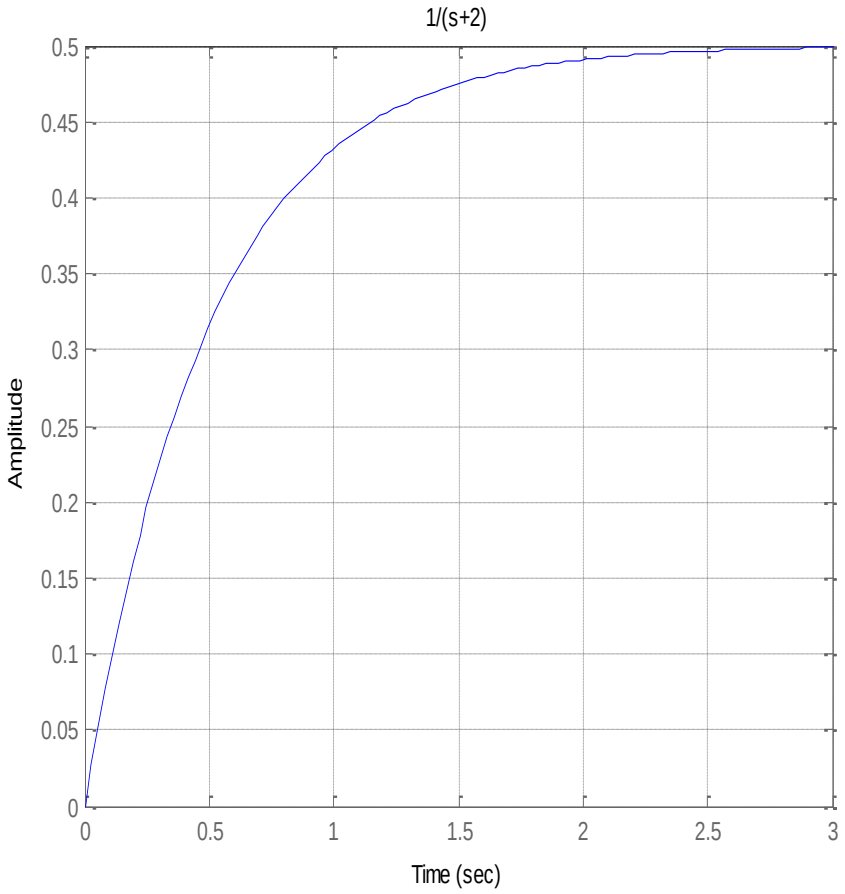
$$step\ \ response$$

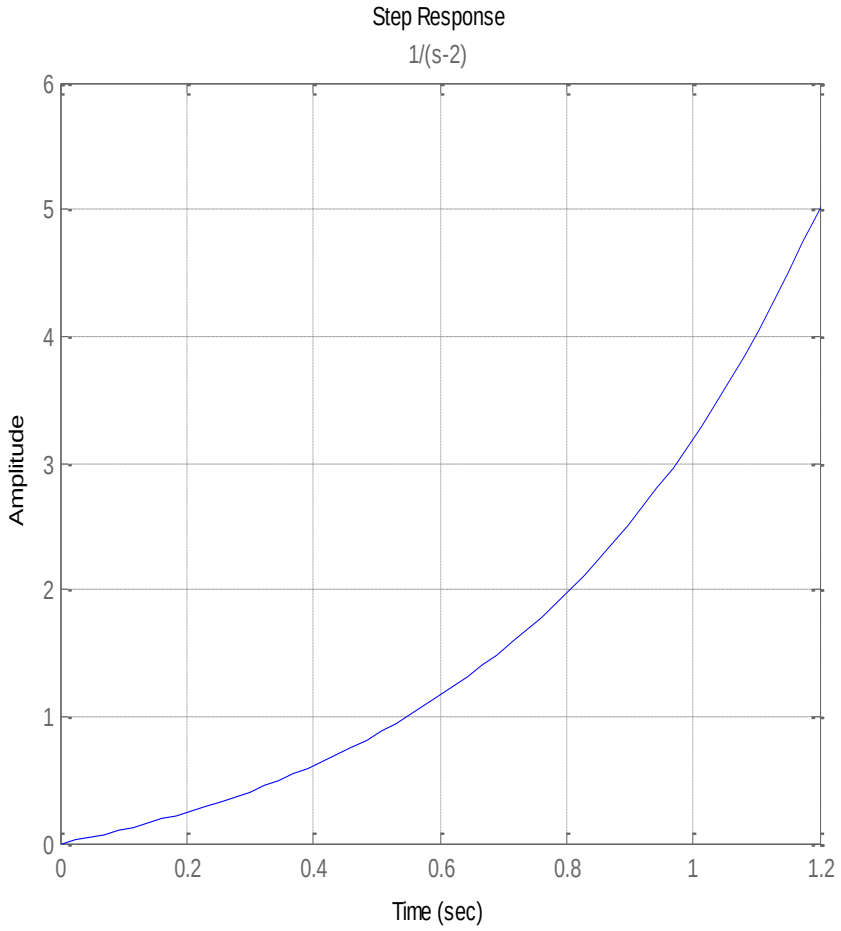
$$y(t) = (1 - e^{-t/T})$$

Time Response

– 1/T > 0, pole is located at s < 0, exponential decays = stable

– 1/T < 0, pole is at s > 0, exponential grows = unstable





4.3 First-Order System

A first-order system (without zero) excited by a unit-step input:

$$C(s) = R(s)G(s) = \frac{1}{s(Ts+1)}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-t/T}$$

when $t = T$: $e^{-t/T}\Big|_{t=T} = e^{-1} = 0.37$

$$c(t) = 1 - 0.37 = 0.63$$

$\therefore T$ is called time constant

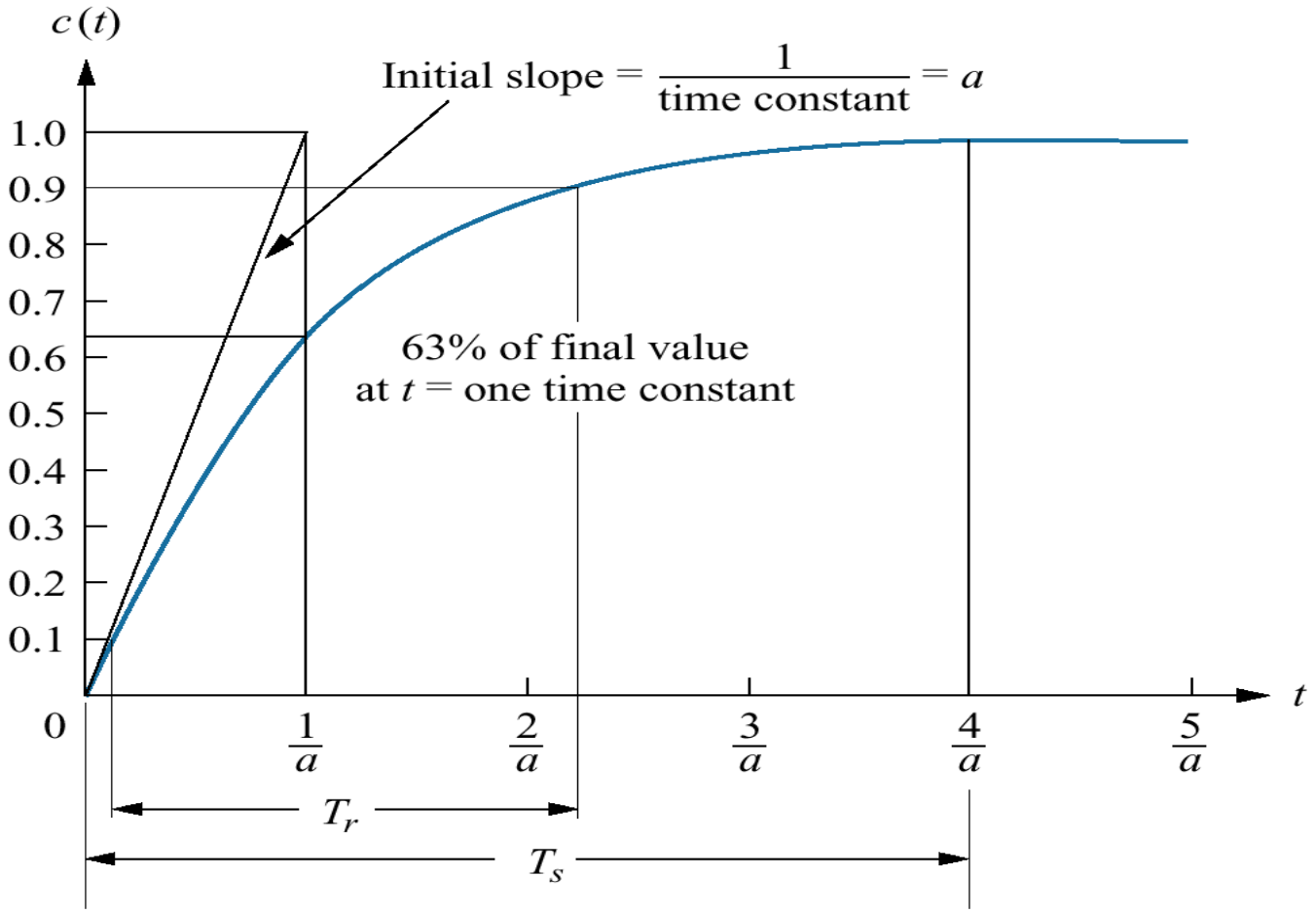
Time for $e^{-t/T}$ to decay to 37%

for to rise to 63% of its final value

$$\frac{1}{T} = exponential\ frequency$$

4.3 First-Order Systems

After a period of one time constant the function reaches e^{-1} = approximately 37% of its initial value



4.3 First order System

First order LTI systems are characterized by the differential equation

$$\frac{dV}{dt} = -\alpha V,$$

where α represents the exponential decay constant and V is a function of time t $V = V(t)$

The time constant is related to the exponential decay constant by

$$\tau = \frac{1}{\alpha}$$

General Solution

The general solution to the differential equation is

$$V(t) = V_o e^{-\alpha t} = V_o e^{-t/\tau}$$

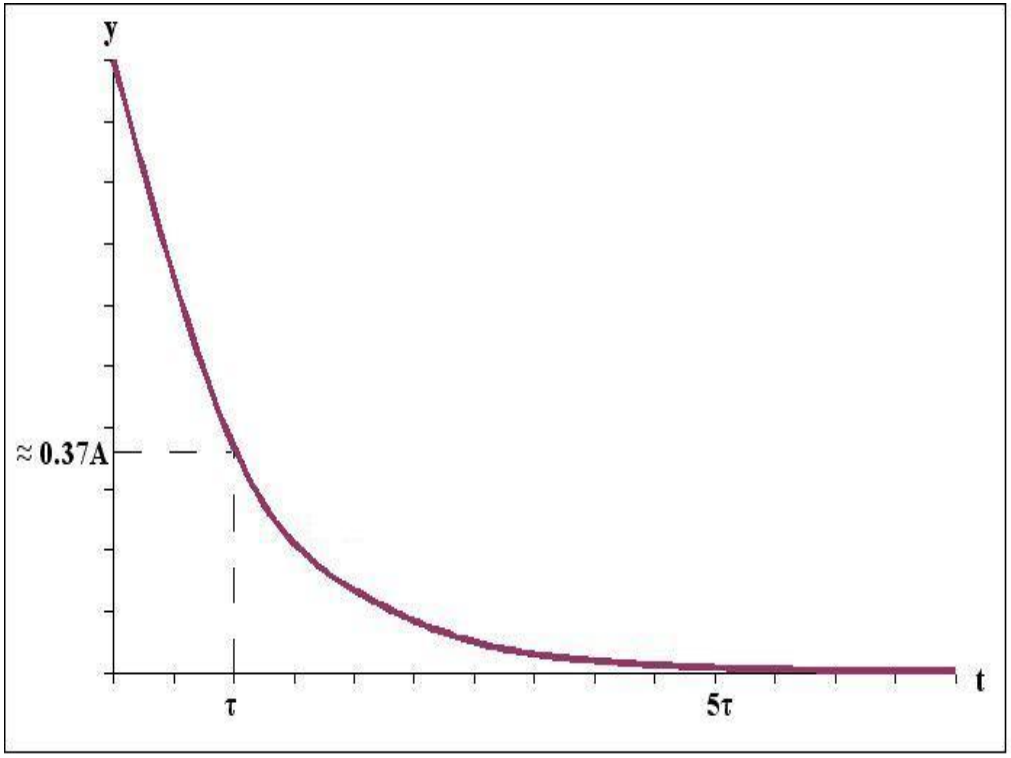
where

$$V_o = V(t=0)$$

is the initial value of V.

4.3 First order System

The diagram below depicts the exponential function $y = Ae^{at}$ in the specific case where $a < 0$, otherwise referred to as a "decaying" exponential function:



Suppose

$$y = Ae^{-at} = Ae^{-\frac{t}{\tau}}$$

then

$$\tau = \frac{1}{a}$$

Specific cases

1). Let $t = 0$, hence $y = Ae^0$, and so $y = A$

2). Let $t = \tau$, hence $y = Ae^{-1}$, $\approx 0.37A$

3). Let $y = f(t) = Ae^{-\frac{t}{\tau}}$, and so $\lim_{t\to\infty} f(t) = 0$

4). Let $t = 5\tau$, hence $y = Ae^{-5}$, $\approx 0.0067A$

4.3 First-Order Systems

• The Rise Time(الزمن المطلوب لزيادة النبضة), T_r, is the time required for the response waveform to go from 0.1 to 0.9 of its final value.

• In the case of the first-order system without zeros.

$$0.9 = 1 - e^{-at_2} \qquad 0.1 = 1 - e^{-at_1}$$

$$t_2 = \frac{-\ln(0.9)}{a} \qquad t_1 = \frac{-\ln(0.1)}{a}$$

$$T_r = \frac{2.3}{a} - \frac{0.1}{a} = \frac{2.2}{a}$$

MATLAB

```
>> t1=-1*log(0.9)

ans =    0.1054

>> t2=-1*log(0.1)

ans = 2.3026
>> Tr=t2-t1
ans= 2.197=2
```

4.3 First-Order Systems

Settling Time(زمن الإستقرار), T_s , is the time required for the response to stabilize within 2% of its final value.

$$0.98 = 1 - e^{-at}$$

$$t = \frac{-\ln(0.02)}{a}$$

$$T_s = \frac{3.9}{a} \approx \frac{4}{a}$$

MATLAB

```
>> t1=-1*log(0.002)

ans =
    3.9120
```

Identifcation of first-order transfer functions:

$$c(t) = \frac{k}{s(s+a)} = \frac{k/a}{s} - \frac{k/a}{s+a}$$

Find k and a :
Time constant

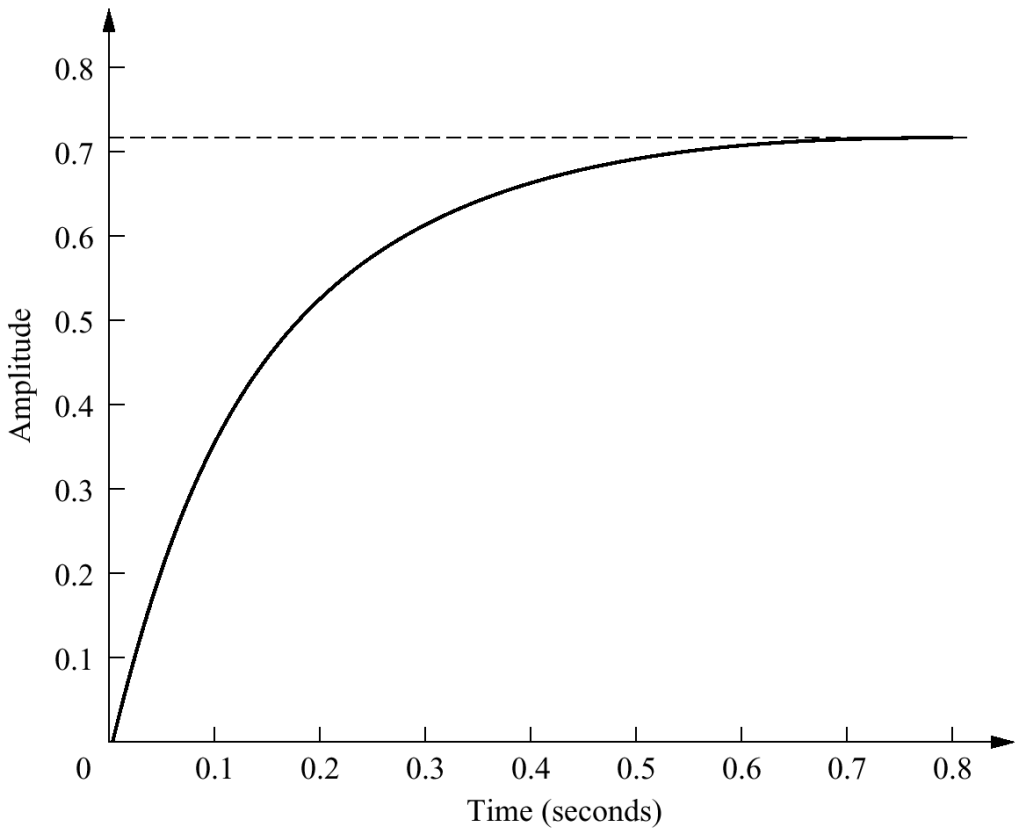
$0.63*0.72=0.45$

$$\frac{1}{a} = 0.13 \rightarrow a = 7.7$$

$$\frac{k}{a} = 7.2 \rightarrow k = 0.72 * 7.7$$

$$= 5.54$$

$$G(s) = \frac{5.54}{s+7.7}$$



4.4 Second-Order Systems: Introduction

- Changing the parameters of a first-order system changes the speed and the final value of the response.

- Changing the parameters of the second-order system changes the speed and the final value of the response as will as the form (shape) of the response.

4.4 Second-Order Systems: Introduction

Second order systems are quite common and are generally written in the following standard form:

$$G(s) = \frac{b}{s^2 + a\,s + b} \quad or$$

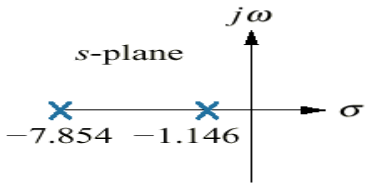
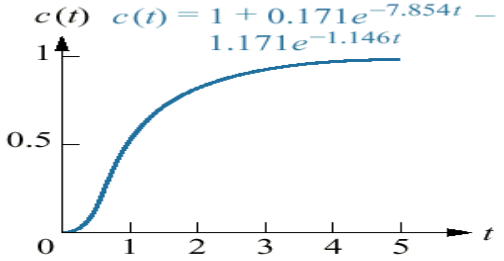
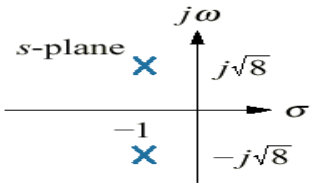
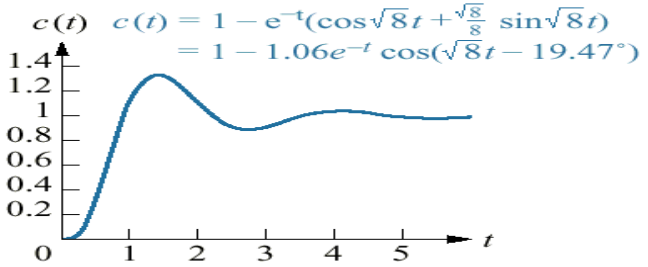
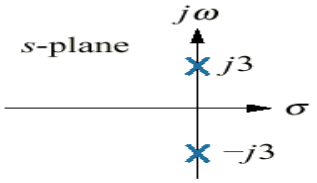
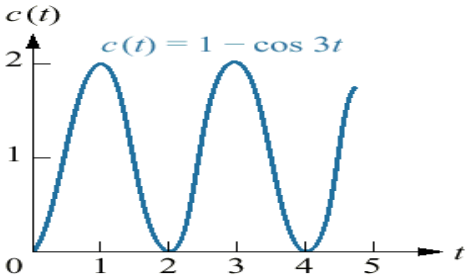
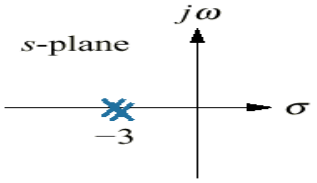
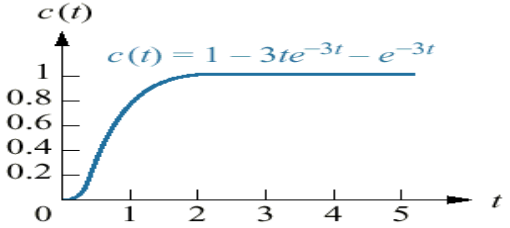
$$G(s) = \frac{\omega_n^2}{s^2 + 2\,\omega_n\,\zeta\,s + \omega_n^2}$$

• The *Natural Frequency*, ω_n, of a second order system is the frequency of oscillation of the system without damping

• The *Damping Ratio*, ζ, of a second-order is the ratio of the exponential decay frequency and the natural frequency.

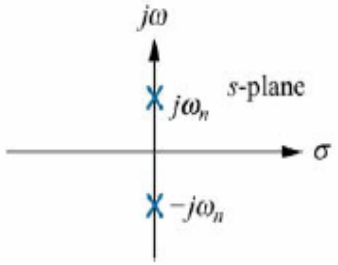
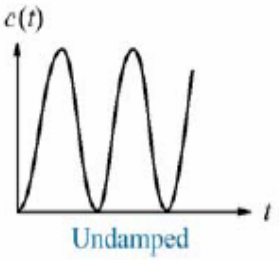
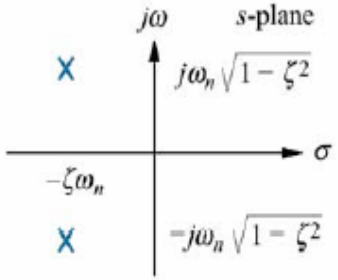
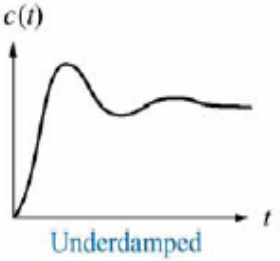
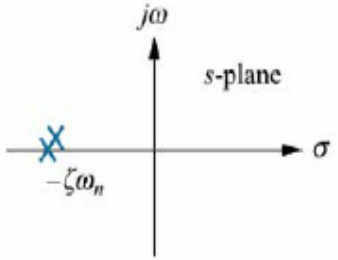
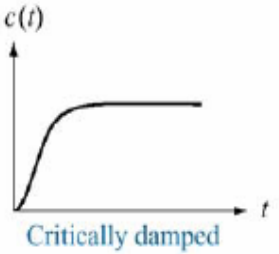
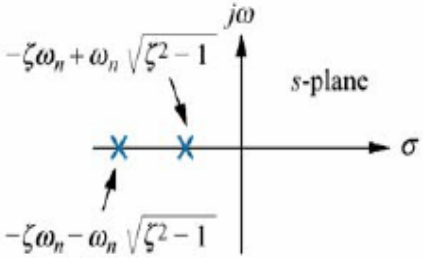
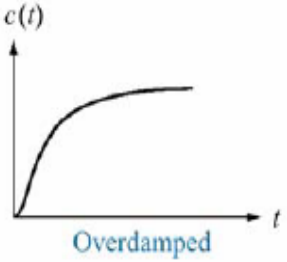
Characteristic equation $s^2 + 2\,\omega_n\,\zeta\,s + \omega_n^2 = 0$

4.4 Second-Order Systems: Introduction

System	Pole-zero Plot	Response
(a) $R(s)=\frac{1}{s}$ → $G(s)=\frac{b}{s^2+as+b}$ → $C(s)$ General		
(b) $R(s)=\frac{1}{s}$ → $G(s)=\frac{9}{s^2+9s+9}$ → $C(s)$ Overdamped		
(c) $R(s)=\frac{1}{s}$ → $G(s)=\frac{9}{s^2+2s+9}$ → $C(s)$ Underdamped		
(d) $R(s)=\frac{1}{s}$ → $G(s)=\frac{9}{s^2+9}$ → $C(s)$ Undamped		
(e) $R(s)=\frac{1}{s}$ → $G(s)=\frac{9}{s^2+6s+9}$ → $C(s)$ Critically damped		

4.5 The General Second-Order Systems

$$G(s)=\frac{\omega_n^2}{s^2+2\omega_n\zeta s+\omega_n^2}$$

ζ	Poles	Step response
0		
$0 < \zeta < 1$		
1		
> 1		

$c(t)=A\cos(\omega_n t-\phi)$

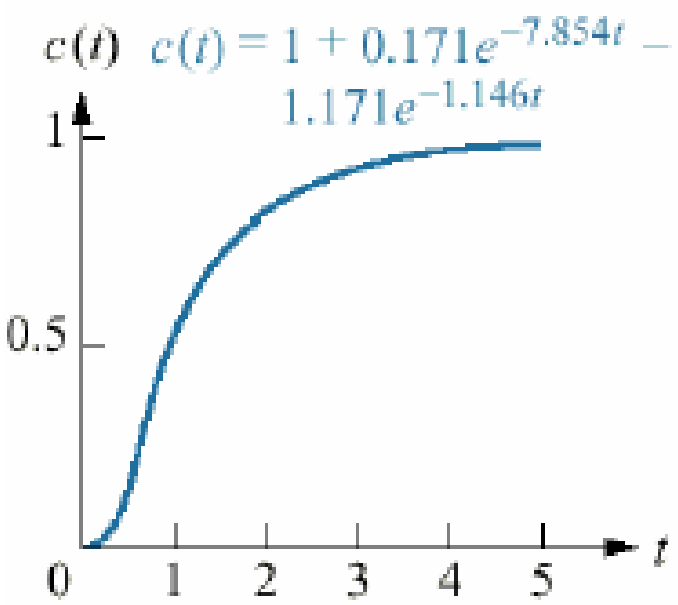
$c(t)=Ae^{-\sigma_d t}\cos(\omega_d t-\phi)$

$c(t)=K_1e^{-\sigma_1 t}+K_2e^{-\sigma_2 t}$

$c(t)=K_1e^{-\sigma_1 t}+K_2te^{-\sigma_1 t}$

4.4 Second-Order Systems: Introduction

Overdamped Response:



$$C(s) = \frac{9}{s(s^2+9s+9)} = \frac{9}{s(s+7.854)(s+1.146)}$$

Poles: pole at the origin $\rightarrow$ constant forced response
Two system poles on the real axis $\rightarrow$ exponential natural response

exponential frequency = pole location

$$c(t) = K_1 + K_2 e^{-7.854t} + k_3 e^{-1.146t}$$

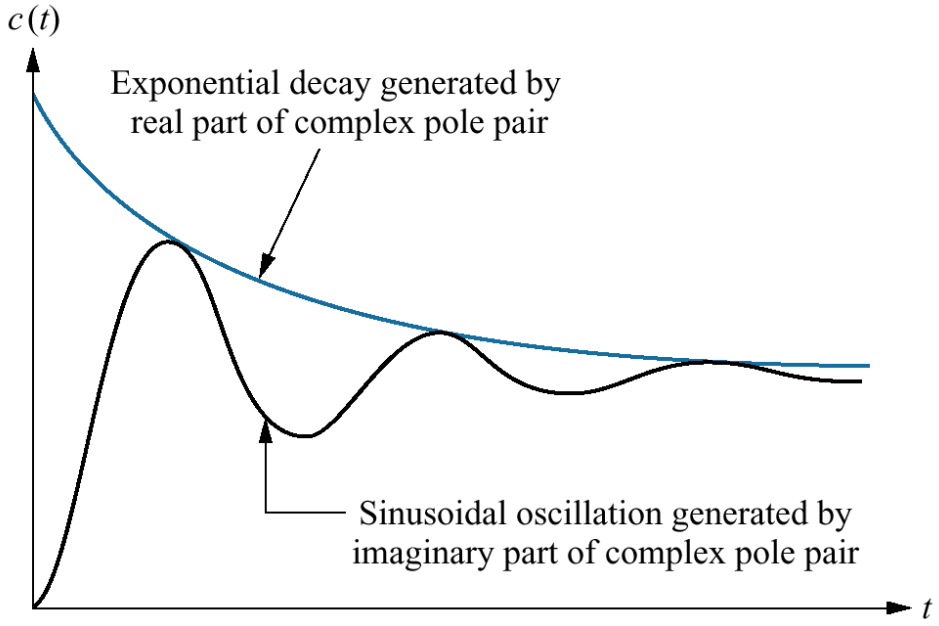
4.4 Second-Order Systems: Introduction

Underdamped Response:

$$C(s) = \frac{9}{s\left(s^2 + 2s + 9\right)}$$

Damped natural frequency: ω_d

Poles: $\sigma_d \pm j\omega_d$ (two complex)

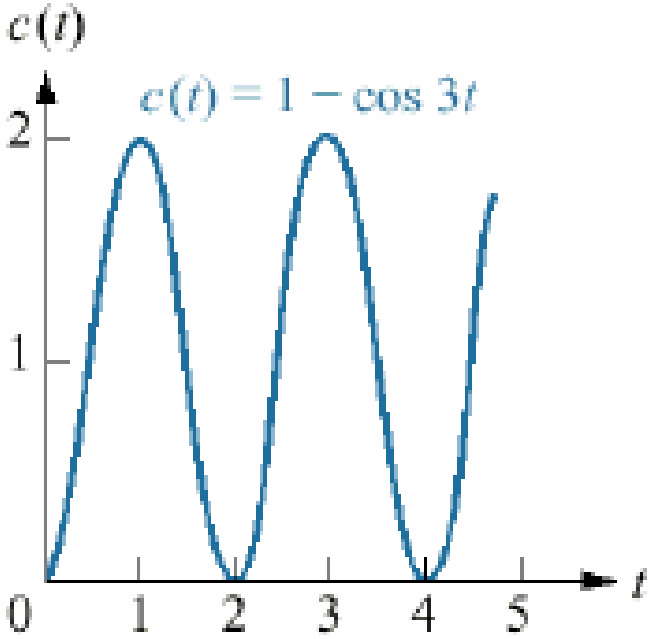


Complex part is responsible for oscillation.

4.4 Second-Order Systems: Introduction

Undamped Response:

$$C(s) = \frac{9}{s(s^2+9)}$$



Poles at the origin $\rightarrow$ constant forced response

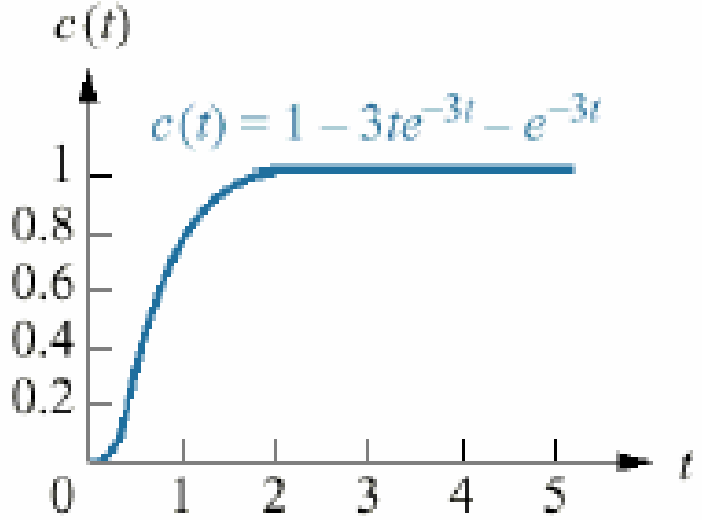
Two poles on the imaginary axis $\rightarrow$ a sinusoidal nautural response with frequency equals to the location of the imaginary poles.

$$c(t) = k_1 + k_2 \cos(3t - \phi)$$

4.4 Second-Order Systems: Introduction

Critically Damped Response:

$$C(s) = \frac{9}{s\left(s^2 + 6s + 9\right)} = \frac{9}{s(s+3)^2}$$



Poles at the origin $\rightarrow$ constant forced response

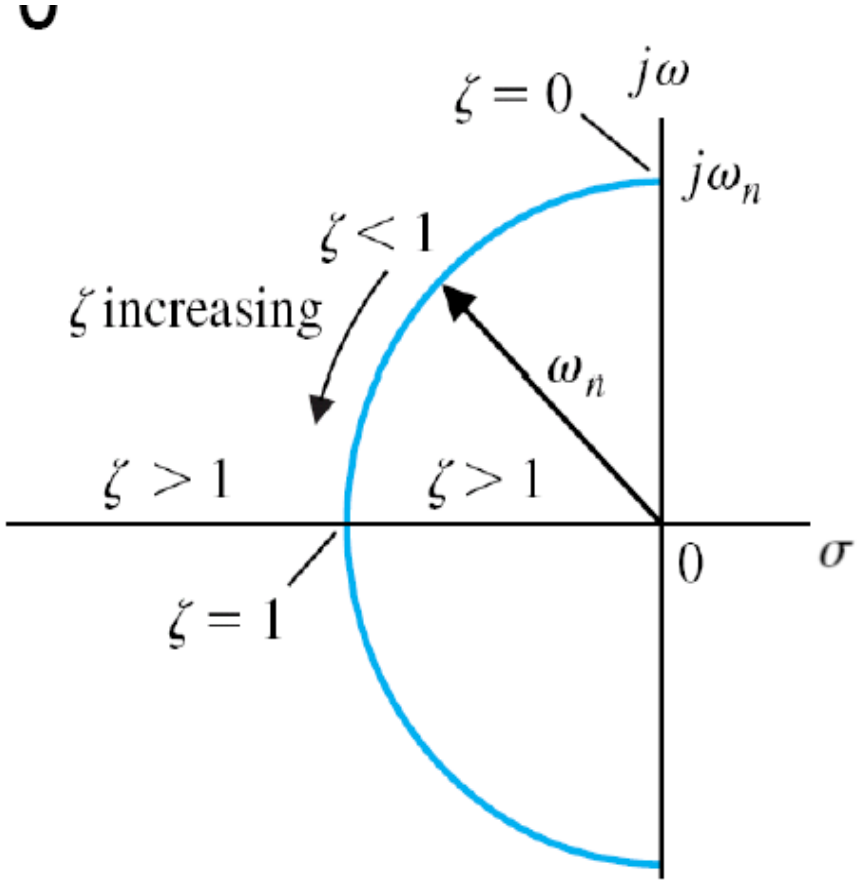
Two poles on the real axis $\rightarrow$ Nautural response is composed of:

1. Exponential
2. Exponential multiplied by time.

Exponential frequency = location of the poles.

$$c(t) = k_1 + k_2 e^{-3t} + k_3 t e^{-3t}$$

4.4 Second-Order Systems: Introduction



4.4 Second-Order Systems: Introduction

Summary

1. Overdamped response :
   Poles: Two real at σ_1 and σ_2
   Natural response : Two exponential with time constant = $\dfrac{1}{\text{pole's location}}$

$$c(t) = k_1 e^{-r_1 t} + k_2 e^{r_2 t}$$

2. Underdamped response
   Poles: two complex at $-\sigma_d \pm j\omega_d$
   Natural response : damped sinusoid with an exponential envolope
   $\left(\text{time constant} = \dfrac{1}{\text{pole's real part}}\right)$,

$$\left(\text{frequency of oscillation= pole's imaginary part}\right)$$

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d - \phi)$$

4.4 Second-Order Systems: Introduction

3. Undamped response:

Poles: two imaginary at $\pm j\omega_1$

Natural response : Undamped sinusoid $\left(\text{fequnency} = \text{imaginary part}\right)$

$$c(t) = A\cos(\omega_1 t - \phi)$$

4. Critically damped response

Poles: two real repeated at $-\sigma_1$

Natural response :

1. Exponential $\left(\text{time constant} = \dfrac{1}{\text{pole location}}\right)$
2. Time multiplied by exponential

$$c(t) = k_1 e^{-\sigma_1 t} + k_2 t e^{-\sigma_1 t}$$

4.5 The General Second Order System

Establish quantitative specifications:

Natural frequency, ω_n: frequency of oscillation without damping.

Damping ratio, ξ: $\dfrac{\text{damping available}}{\text{damping necessary for critical damping}}$

$$\xi = \frac{\textit{Exponential decay frequency}}{\textit{Natural frequency}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$

$$= \frac{1}{2\pi}\frac{\textit{Natural period}}{\textit{Exponential time constant}}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}\zeta\right)/\sqrt{1-\zeta^2}$$

"time constant"

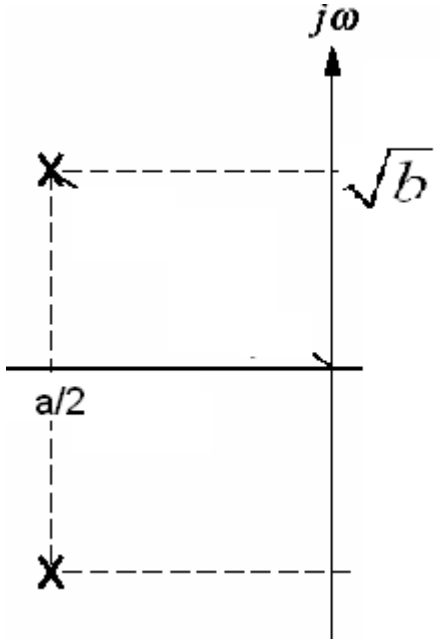
damping ratio

natural frequency

oscillation/vibration frequency

4.5 The General Second-Order Systems

$$G(s) = \frac{b}{s^2 + as + b}$$



Without damping

$$G(s) = \frac{b}{s^2 + b}$$

$$\omega_n = \sqrt{b} \text{ or } b = \omega_n{}^2$$

Response : $A\sin\left(\sqrt{b}t\right)$

With damping

$$\xi = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}}$$

$$\xi = \frac{a/2}{\omega_n} \text{ or } a = 2\xi\omega_n$$

$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}$$

4.5 The General Second-Order Systems

• How does this relate to the pole location in terms of their real and imaginary parts?

$$s = -\sigma \pm j\,\omega_d$$

ω_d Damped freq. of oscillation

Complex poles are always in complex conjugate pairs so:

$$d(s) = (s - (-\sigma + j\omega_d))\ (s - (-\sigma - j\omega_d))$$

$$d(s) = (s + \sigma + j\omega_d)\ (s + \sigma - j\omega_d)$$

$$d(s) = (s + \sigma)^2 + \omega_d{}^2$$

4.5 The General Second-Order Systems

• By multiplying out and comparing the denominator of the two forms we find that

$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n \zeta s + \omega_n^2}$$

$$d(s) = (s+\sigma)^2 + \omega_d^2$$

$$s^2 + 2\omega_n \zeta s + \omega_n^2 = (s+\sigma)^2 + \omega_d^2$$

4.5 The General Second-Order Systems

• By multiplying out and comparing the denominator of the two forms we find that

$$s^2 + 2\omega_n \zeta s + \omega_n^2 = (s+\sigma)^2 + \omega_d^2$$

$$\sigma = \zeta \omega_n$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

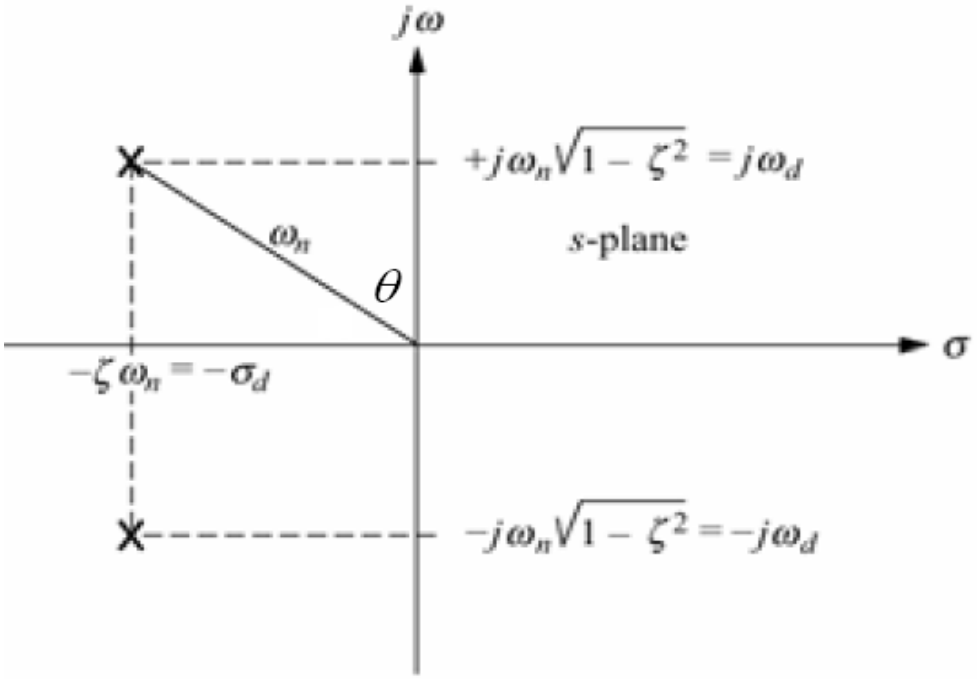
4.5 The General Second-Order Systems

• We can now relate the natural frequency and damping ratio to the *s*-plane

$$\sigma = \zeta \omega_n$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\theta = \cos^{-1}(\zeta)$$



4.5 The General Second-Order Systems

Example

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

$$\omega_n{}^2 = 36 \rightarrow \omega_n = 6$$

$$2\xi\omega_n = 4.2 \rightarrow \xi = \frac{4.2}{2*6} = 0.35$$

Since $0 < \xi < 1$

$\therefore$ Underdamped response

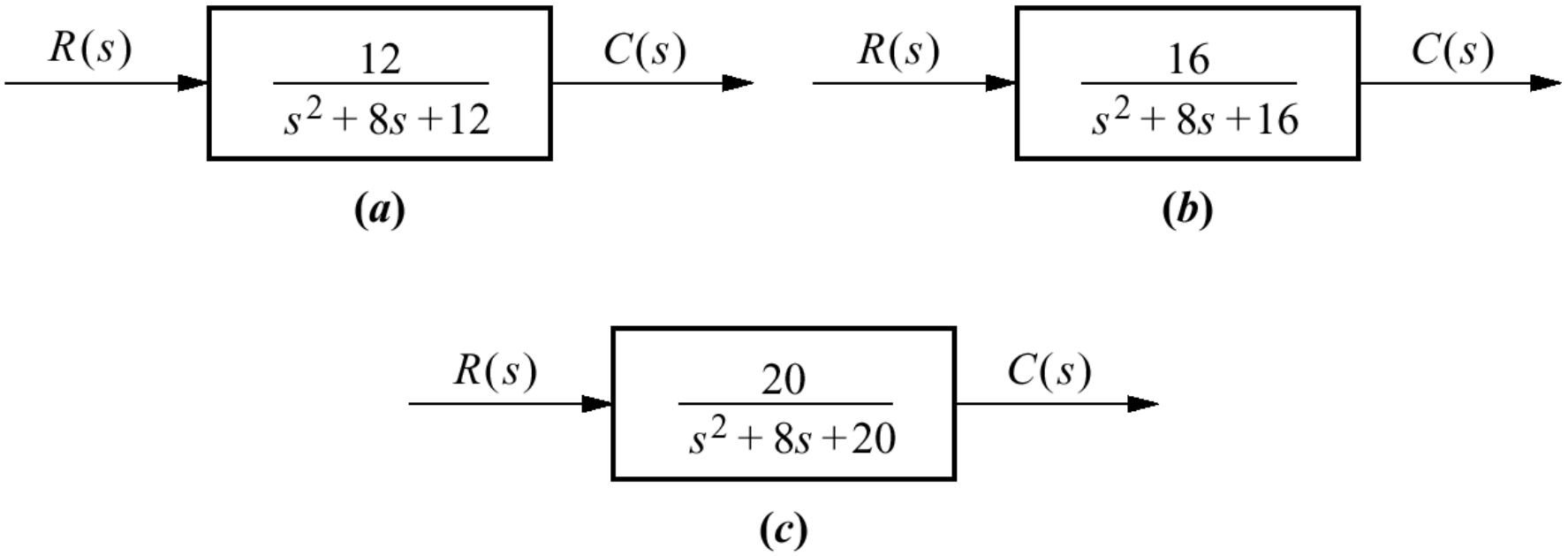
$$G(s) = \frac{\omega_n{}^2}{s^2 + 2\xi\omega_n s + \omega_n{}^2}$$

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

4.5 The General Second-Order Systems

Example

Find ξ and determine the type of response



4.6 Underdamped Second-Order Systems

$$R(s) = \frac{1}{s}, \quad \xi < 1$$

$$C(s) = \frac{\omega_n^{\,2}}{s\left(s^2 + 2\xi\omega_n s + \omega_n^{\,2}\right)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}$$

$$C(s) = \frac{1}{s} - \frac{(s + \xi\omega_n) + \dfrac{\xi}{\sqrt{1-\xi^2}}\,\omega_n\sqrt{1-\xi^2}}{(s + \xi\omega_n)^2 + \omega_n^{\,2}\left(1-\xi^2\right)}$$

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\cos\left(\omega_n\sqrt{1-\xi^2} * t - \phi\right)$$

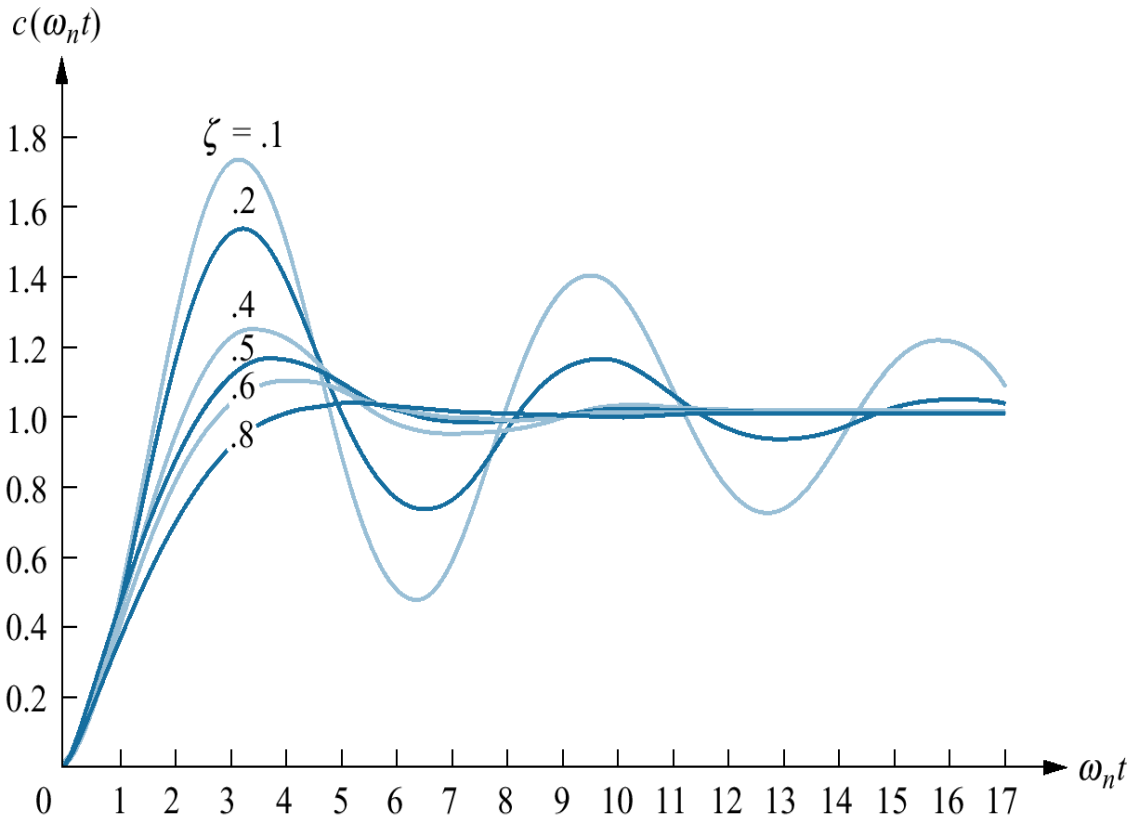
where

$$\phi = \tan^{-1}\left(\xi \Big/ \sqrt{1-\xi^2}\right)$$

4.6 Under Damped Second-Order Systems

- The damping ratio describes the degree of damping in the System

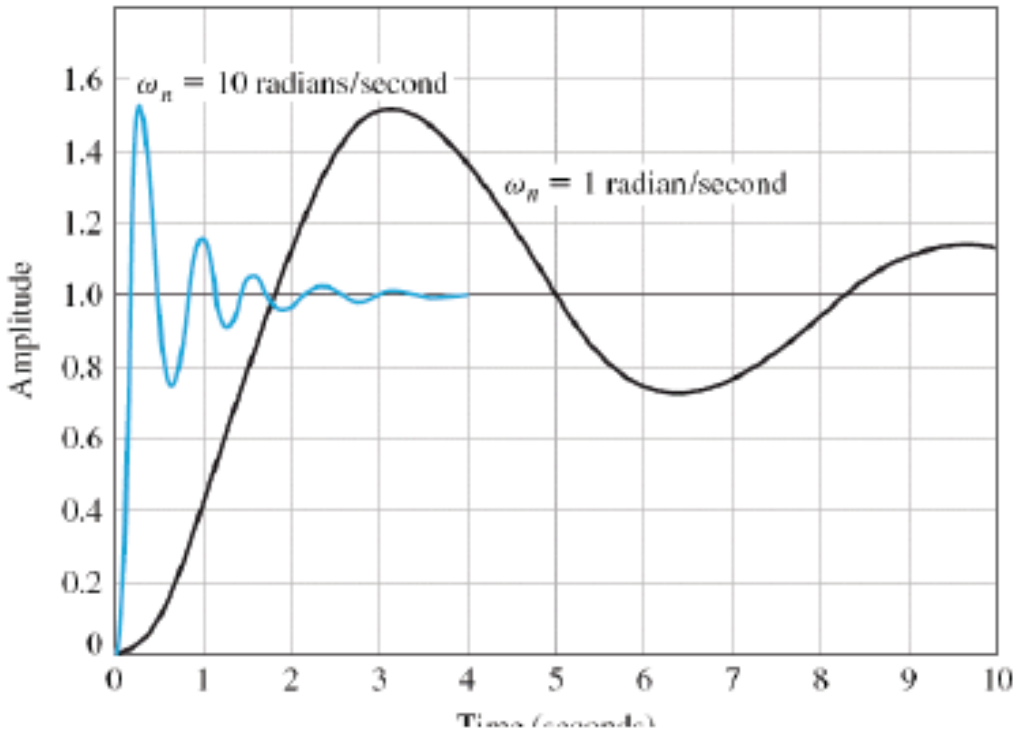
- For an underdamped system, rising damping will lower the overshoot of the system



4.6 Under Damped Second-Order Systems

Effect of ω_n

Frequency of oscillations



4.6 Under Damped Second-Order Systems

Specifications for a control system design often involve equirements associated with (مُرْتَبِط)the time response of the system

- *Rise time*, T_r: The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.

  (is the time it take for the system to reach the vicinity(قُرْب) of its new set point)

- *Peak time*, T_p: The time required to reach the first, maximum, peak.

  (is the time taken to reach maximum overshoot)

- *Present overshoot*, *%OS*: The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.

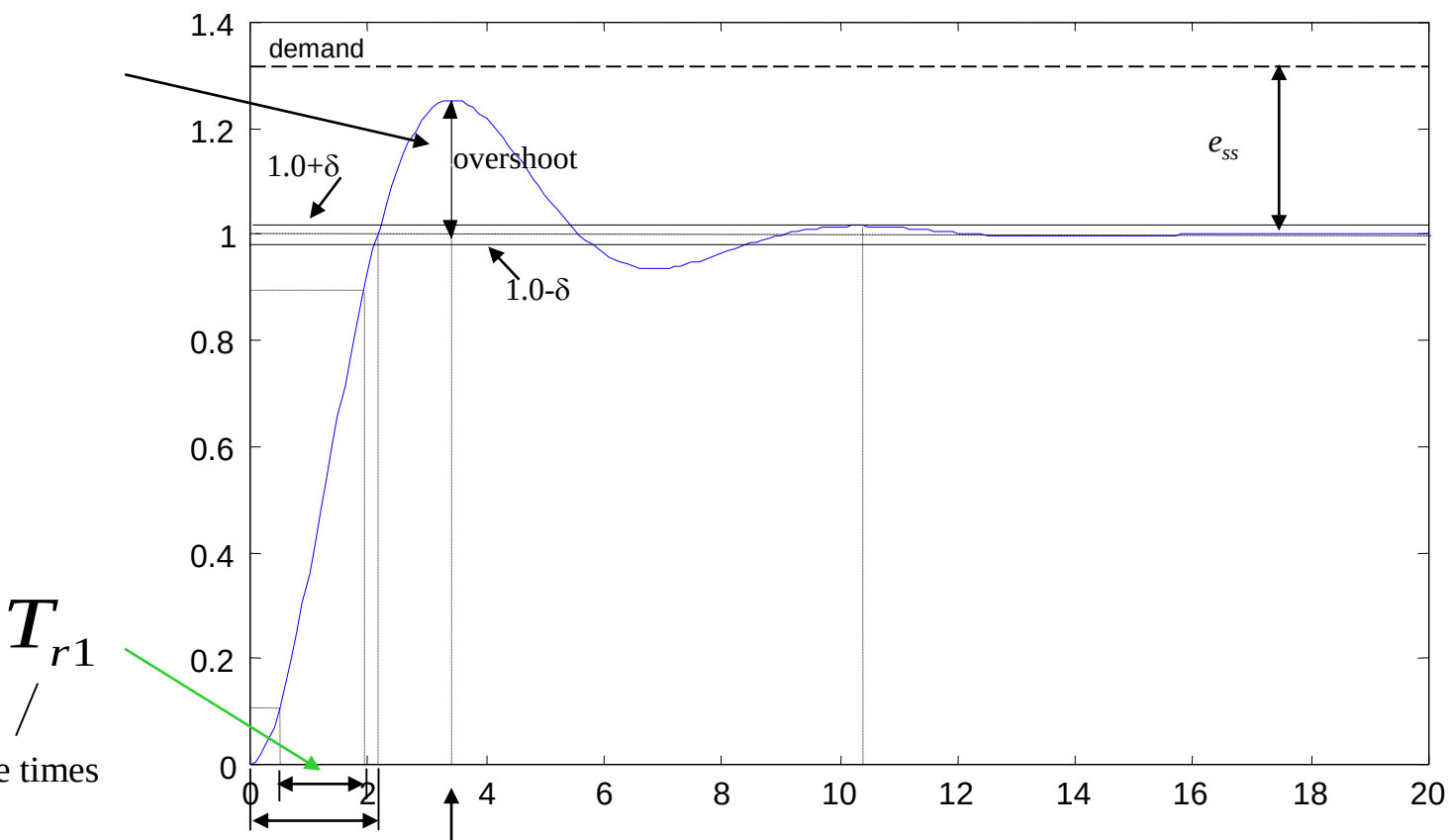
  (is the maximum amount the system overshoots its final value)

- *Settling time*, T_s: The time required for the transient's damped oscillations to reach and stay within ±2% of the steady-state value.

  (is the time for system transients to decay)

4.6 Under Damped Second-Order Systems

%OS = percentage overshoot



T_{r1}

rise times

T_r

T_p

peak time

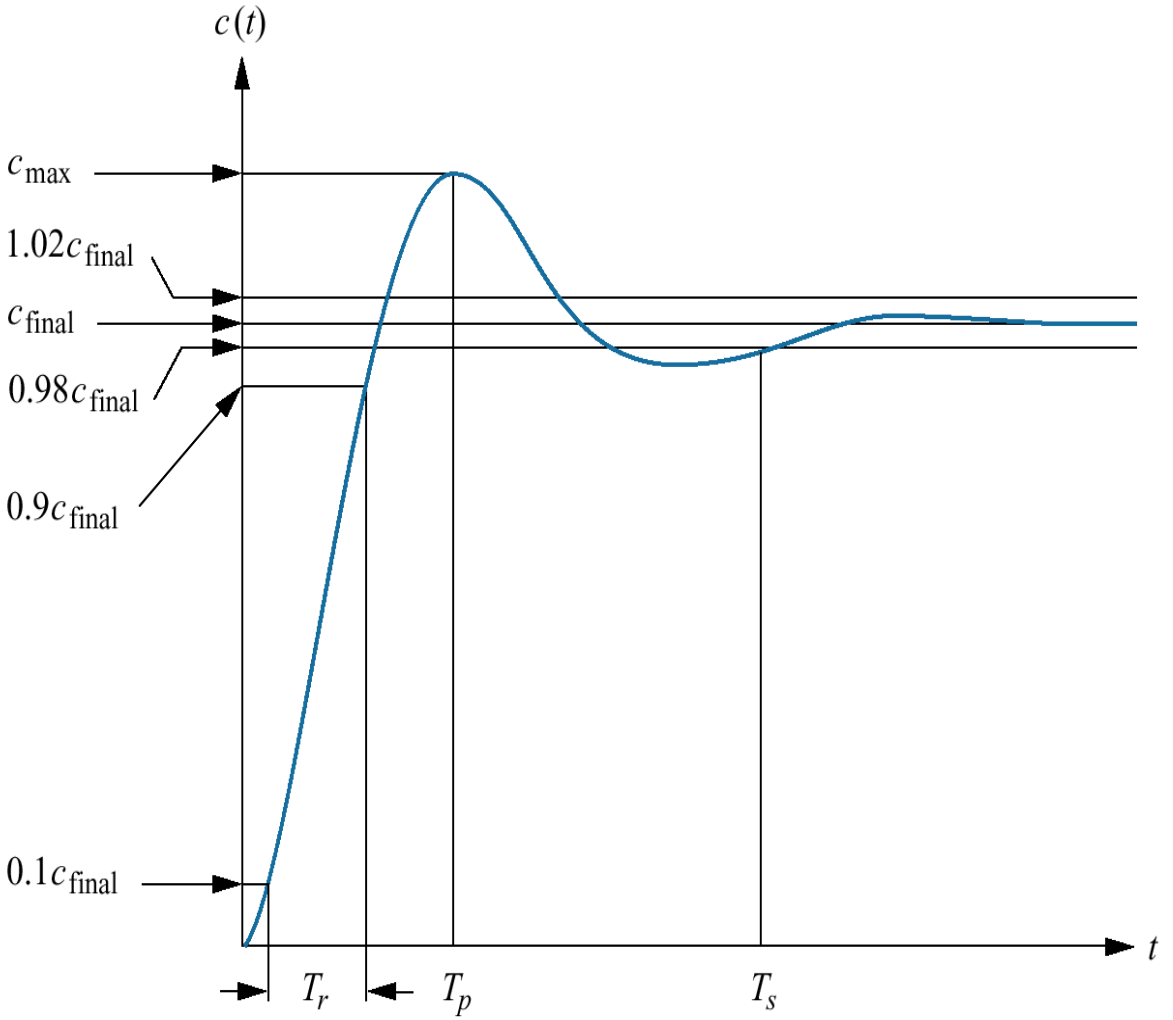
T_s

settling time for δ (usually corresponding to 1-2%)

4.6 Under Damped Second-Order Systems

- Rise time, settling time and peak time yield information about the speed of response of the transient response

- This can help a designer determine if the speed and nature of the response is appropriate(مُلَائِم)



4.6 Under Damped Second-Order Systems

Evaluation of T_p:

- Peak time is found by differentiating and finding the first zero crossing after t=0

- After some manipulation we find t_p occurs when $\sin \omega_d t = 0$

$$\mathcal{L}\left[c(t)\right] = sC(s) = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}}$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} * t\right)$$

$$c(t) = 0 \rightarrow \omega_n \sqrt{1-\xi^2}\, t = n\pi$$

$$t = \frac{n\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

4.6 Under Damped Second-Order Systems

Evaluation of %*OS*:

- Substituting the peak time back into the step response yields

$$\%OS = \frac{c_{\max} - c_{final}}{c_{final}} * 100$$

$$c_{\max} = 1 + e^{-\left(\xi\pi / \sqrt{1-\xi^2}\right)}$$

for $c_{final} = 1$

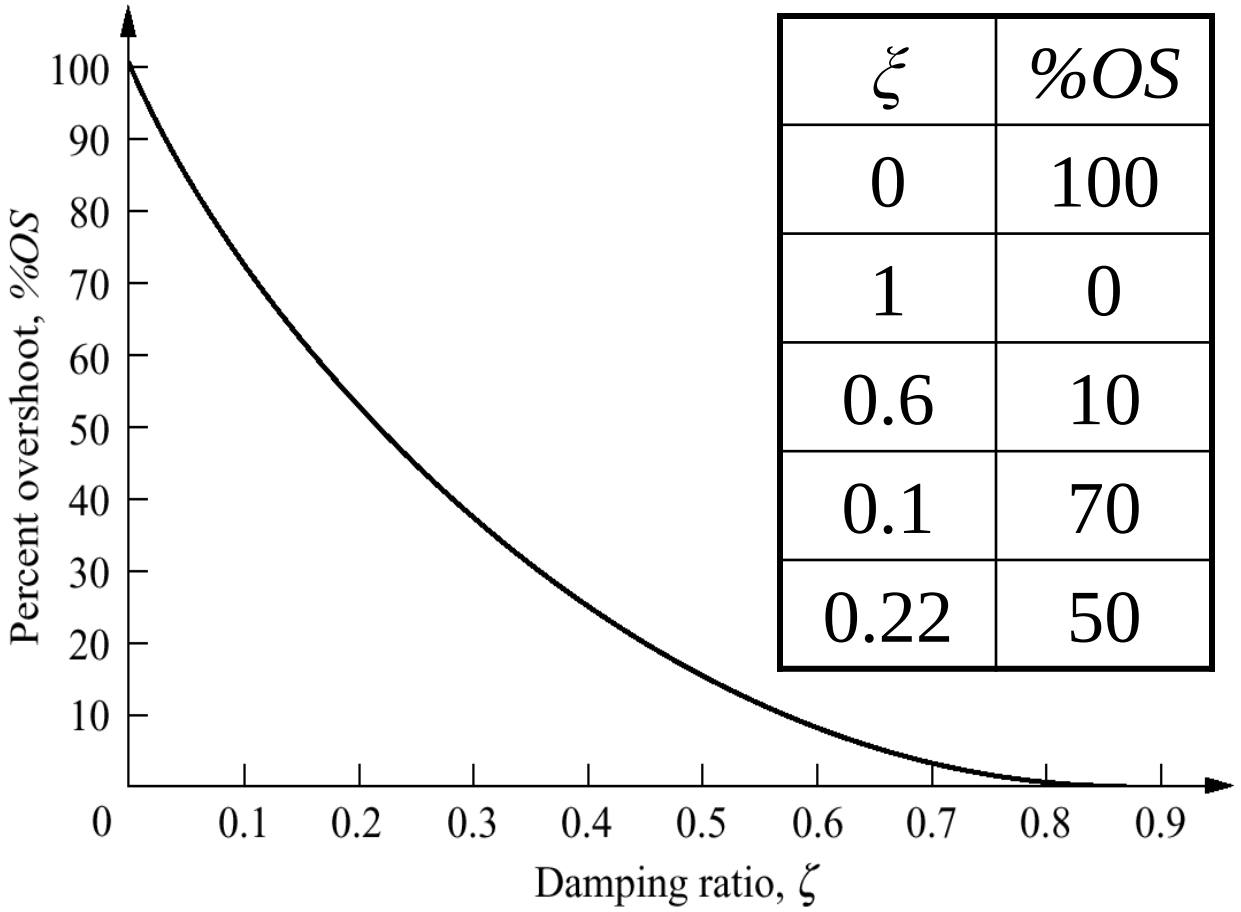
- Notice this is only a function of the damping ratio. We can invert to find the required damping ratio for a particular overshoot

$$\%OS = e^{-\left(\xi\pi / \sqrt{1-\xi^2}\right)} * 100 \Rightarrow \xi = \frac{-\ln\left(\%OS/100\right)}{\sqrt{\pi^2 + \ln^2\left(\%OS/100\right)}}$$

4.6 Under Damped Second-Order Systems

• We can plot the relationship between percent overshoot and damping ratio

• Two frequently used specifications are
– Mp=5% for ζ=0.7
– Mp=16% for ζ=0.5



ξ	%OS
0	100
1	0
0.6	10
0.1	70
0.22	50

4.6 Under Damped Second-Order Systems

Evaluation of T_s:

- The settling time can be derived in a similar manner to for a first order system

$$c(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cos\left(\omega_n \sqrt{1-\xi^2} * t - \phi\right)$$

$$e^{-\xi\omega_n t} \frac{1}{\sqrt{1-\xi^2}} = 0.02$$

- The settling time is essentially decided by the transient exponential

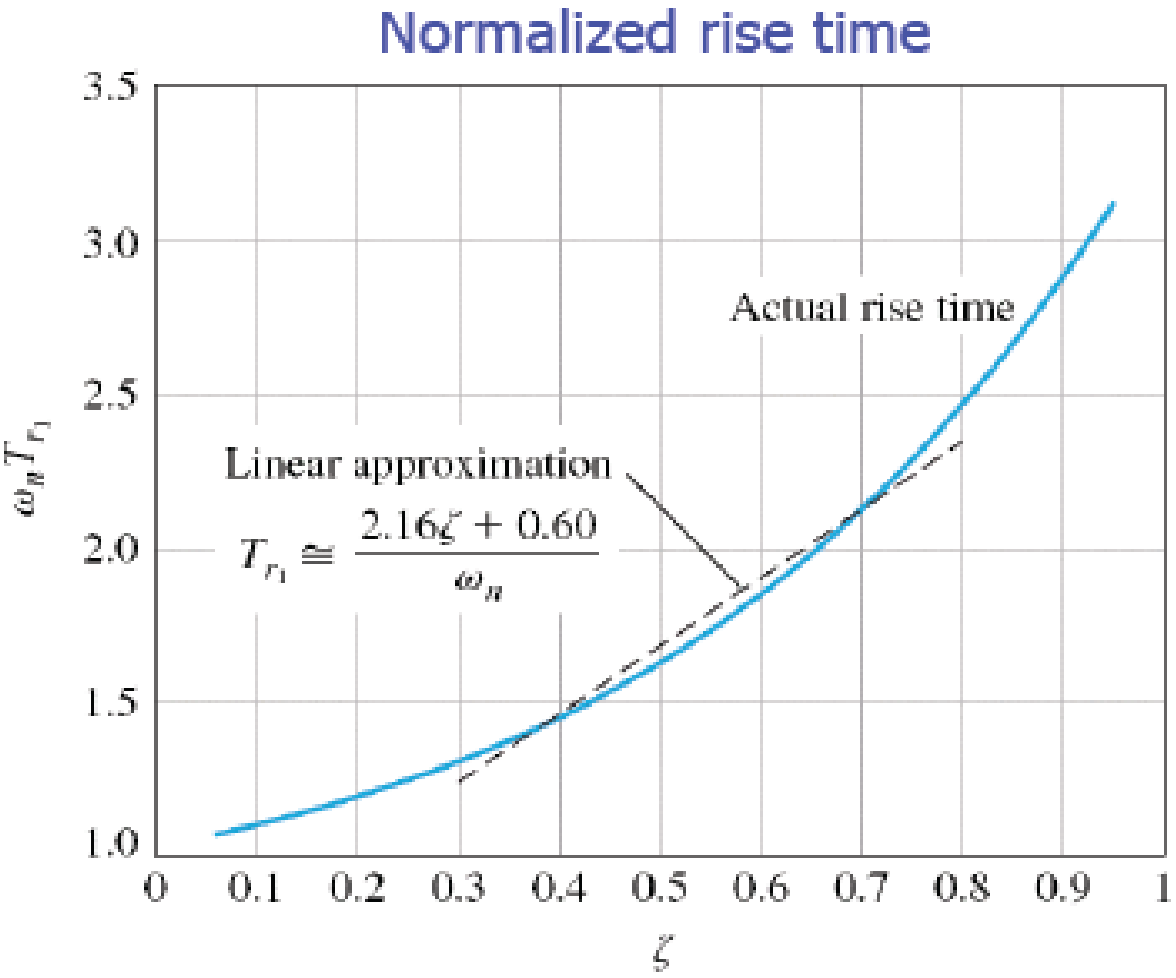
$$T_s = \frac{-\ln(0.02\sqrt{1-\xi^2})}{\xi\omega_n}$$

$$T_s \approx \frac{4}{\xi\omega_n}$$

$$T_s \approx 4 \text{ time constant}$$

4.6 Under Damped Second-Order Systems

Evaluation of T_r:



$$T_r = \frac{2.16\xi + 0.60}{\omega_n}$$

No explicit solution in terms of ξ and ω_n

4.6 Under Damped Second-Order Systems

Example

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

Find $T_p, \%OS, T_s$ and T_r

$$\omega_n = 10$$

$$2\xi\omega_n = 15 \rightarrow \xi = \frac{15}{20} = 0.75$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.475\,\text{sec.}$$

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.75\text{x}10} = 0.533\,\text{sec.}$$

$$\%OS = e^{-\left(\xi\pi/\sqrt{1-\xi^2}\right)} * 100 = 2.83$$

$$T_r\omega_n = 2.3 \rightarrow T_r = \frac{2.3}{10} = 0.23\,\text{sec}$$

4.6 Under Damped Second-Order Systems

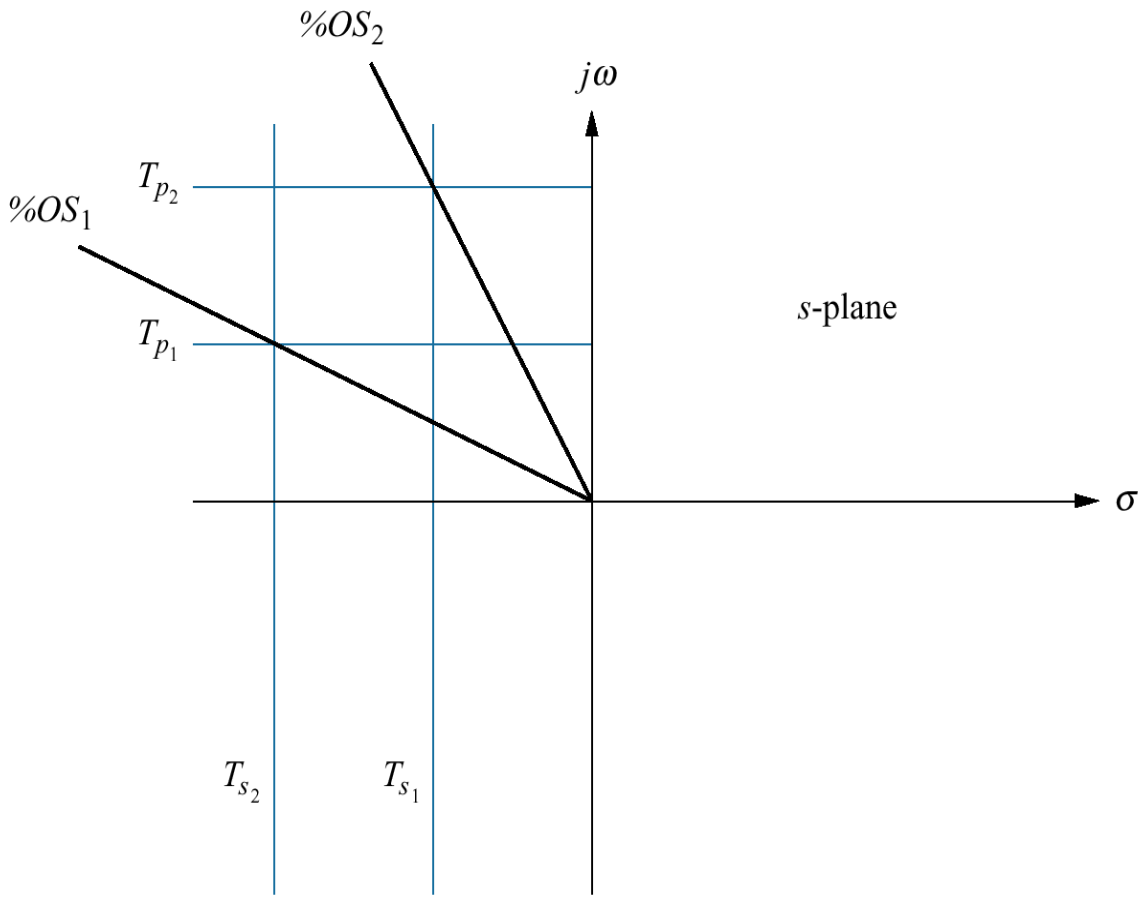
• Lines of constant peak time, Tp , settling time, Ts , and percent overshoot, %OS and rise time

• Note:

$Ts_2 < Ts_1$;

$Tp_2 < Tp_1$;

$\%OS_1 < \%OS_2$



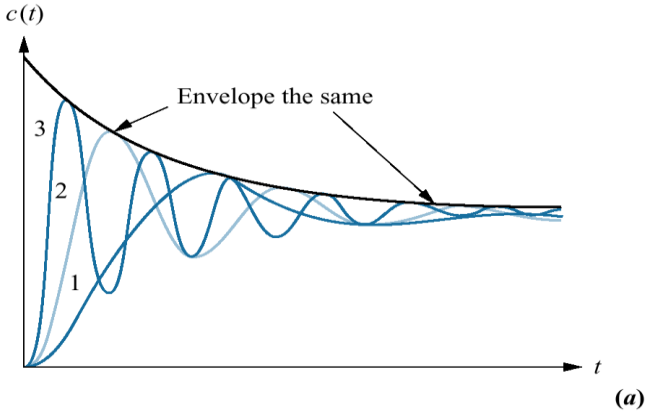
4.6 Under Damped Second-Order Systems

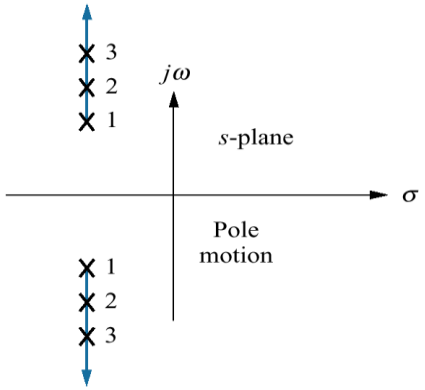
• Step responses of second-order underdamped Systems as poles move:

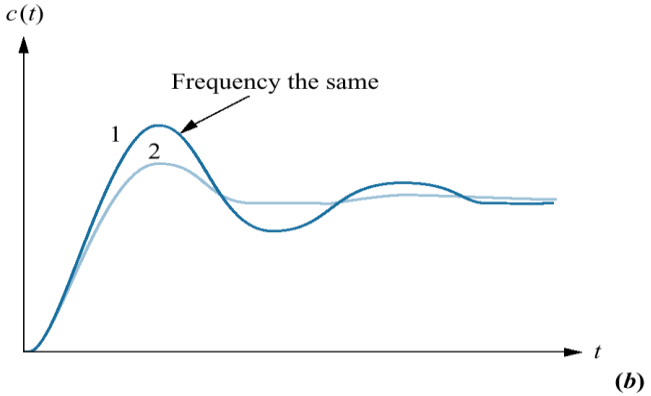
a. with constant real part σ;

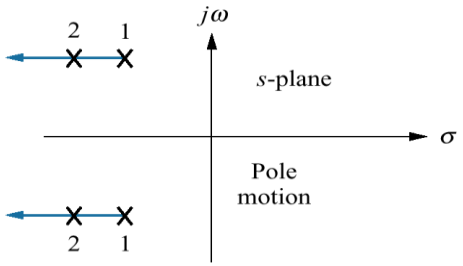
b. with constant imaginary part jω;

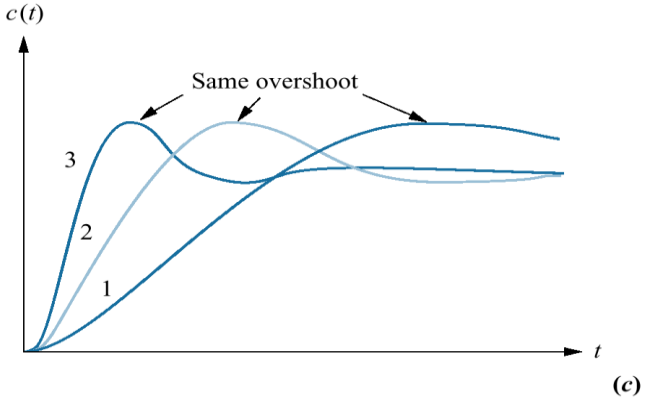
c. with constant damping ratio ζ

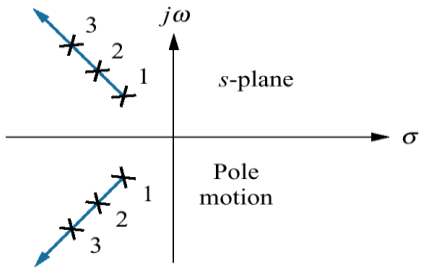






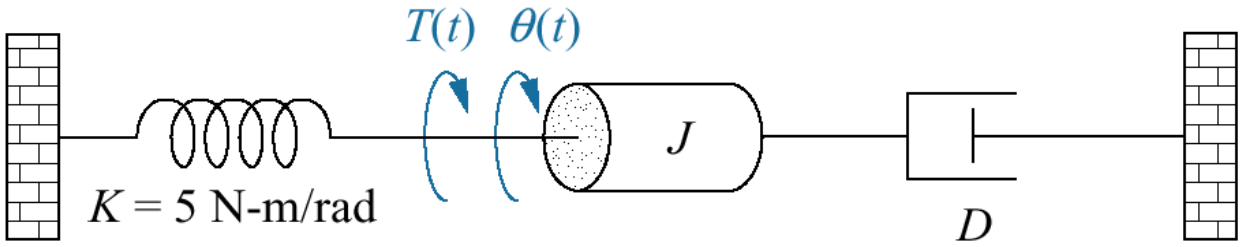






4.6 Under Damped Second-Order Systems

Example 4.7



Find J, D: $20\%overshoot, T_s = 2$ *sec.* for step torque input.

$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$T_s = 2 = \frac{4}{\xi\omega_n} \rightarrow \xi\omega_n = 2$$

$$20\%OS \rightarrow \xi = 0.456$$

$$\omega_n{}^2 = \frac{K}{J},\ K = 5$$

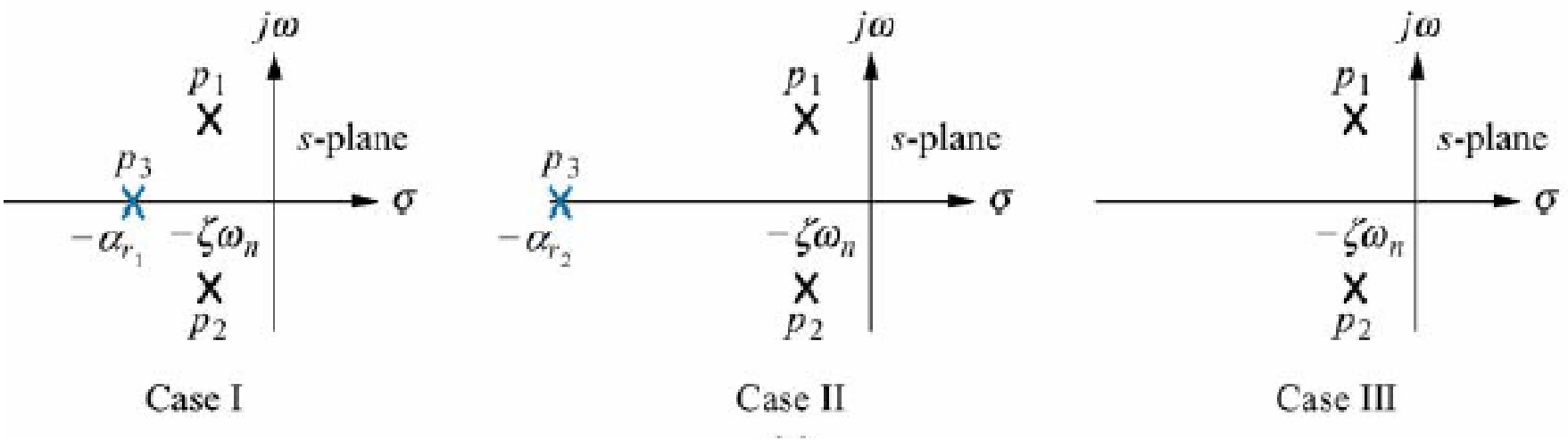
$$\omega_n{}^2 = \frac{K}{J},\ K = 5$$

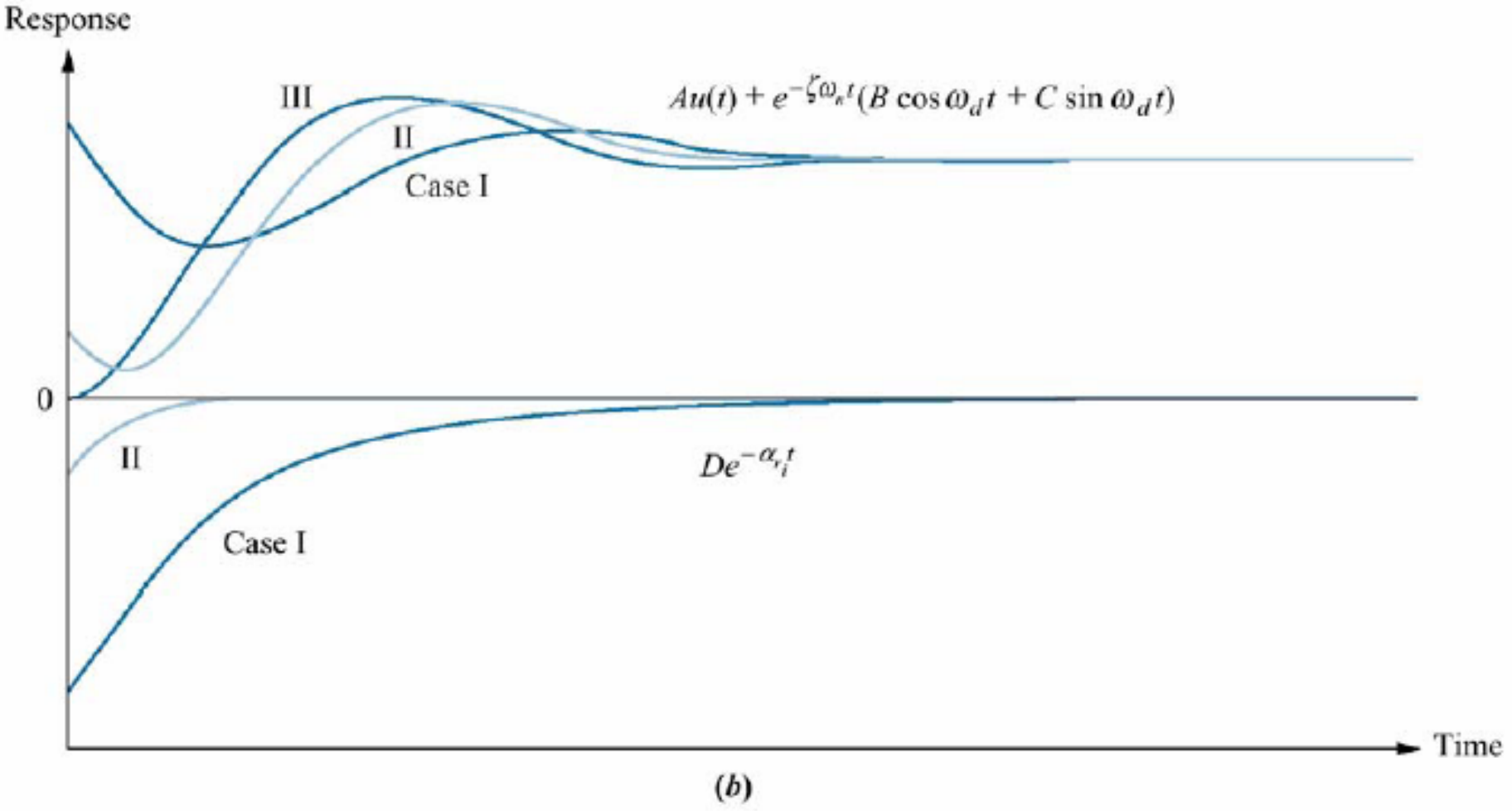
$$2\xi\omega_n = D/J \qquad \therefore D = 1.04\ N.m.s/rad \qquad J = 0.26\ kg.m^2$$

4.7 System Response with Additioal Pole

$$C(s) = \frac{A}{s} + \frac{B(s+\xi\omega_n) + C\omega_d}{(s+\xi\omega_n)^2 + \omega_d{}^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\xi\omega_n t}\left(B\cos(\omega_d t) + C\sin(\omega_d t)\right) + De^{-\alpha_r t}$$



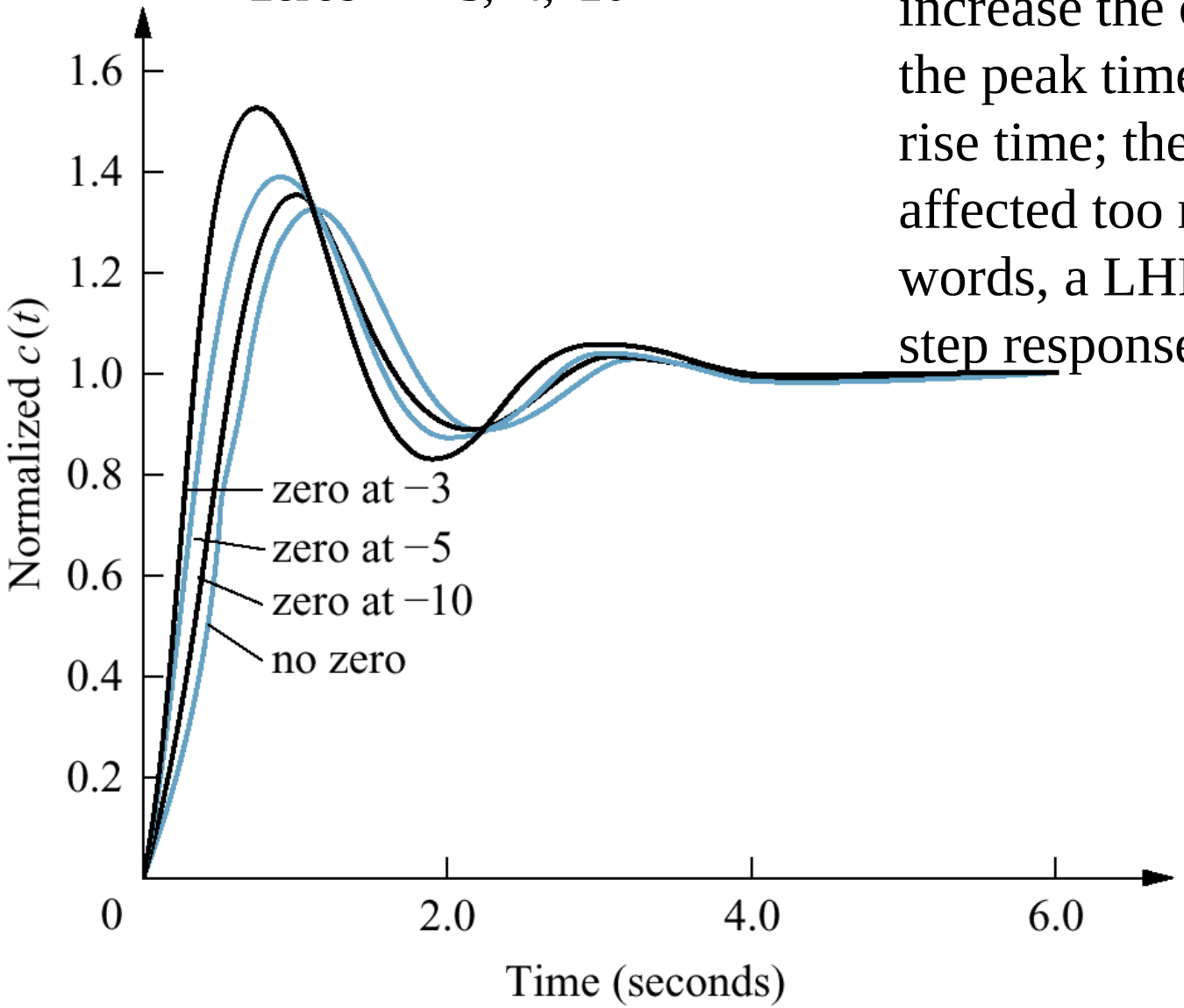


If the real pole is 5 times farther to the left than the dominant poles: dominant poles

4.8 System Response with Zeros

$$\text{Poles} = -1 \pm j2.828$$

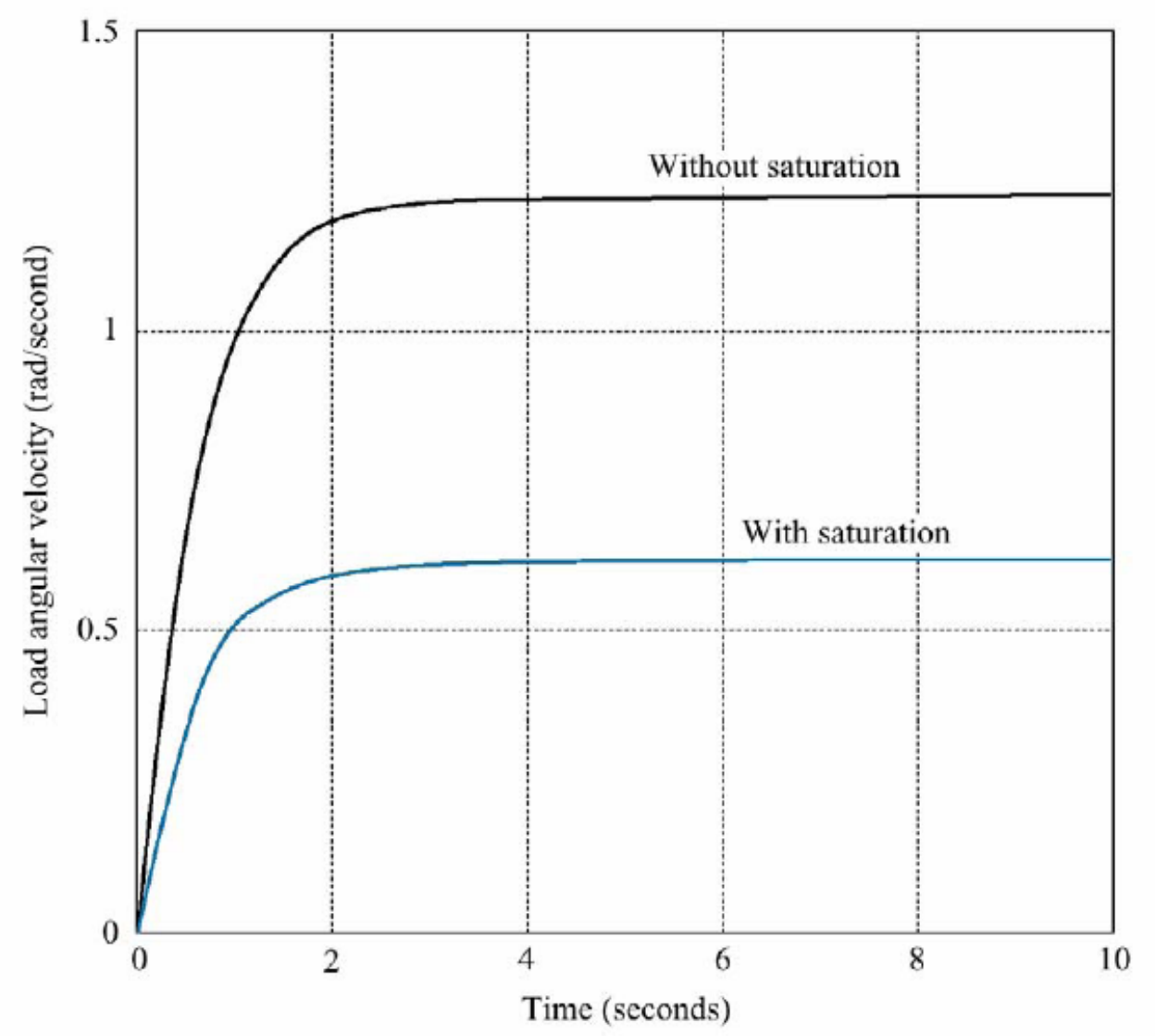
$$\text{zeros} = -3, -4, -10$$



The effect of a LHP zero is to increase the overshoot, decrease the peak time, and decrease the rise time; the settling time is not affected too much. In other words, a LHP zero makes the step response faster.

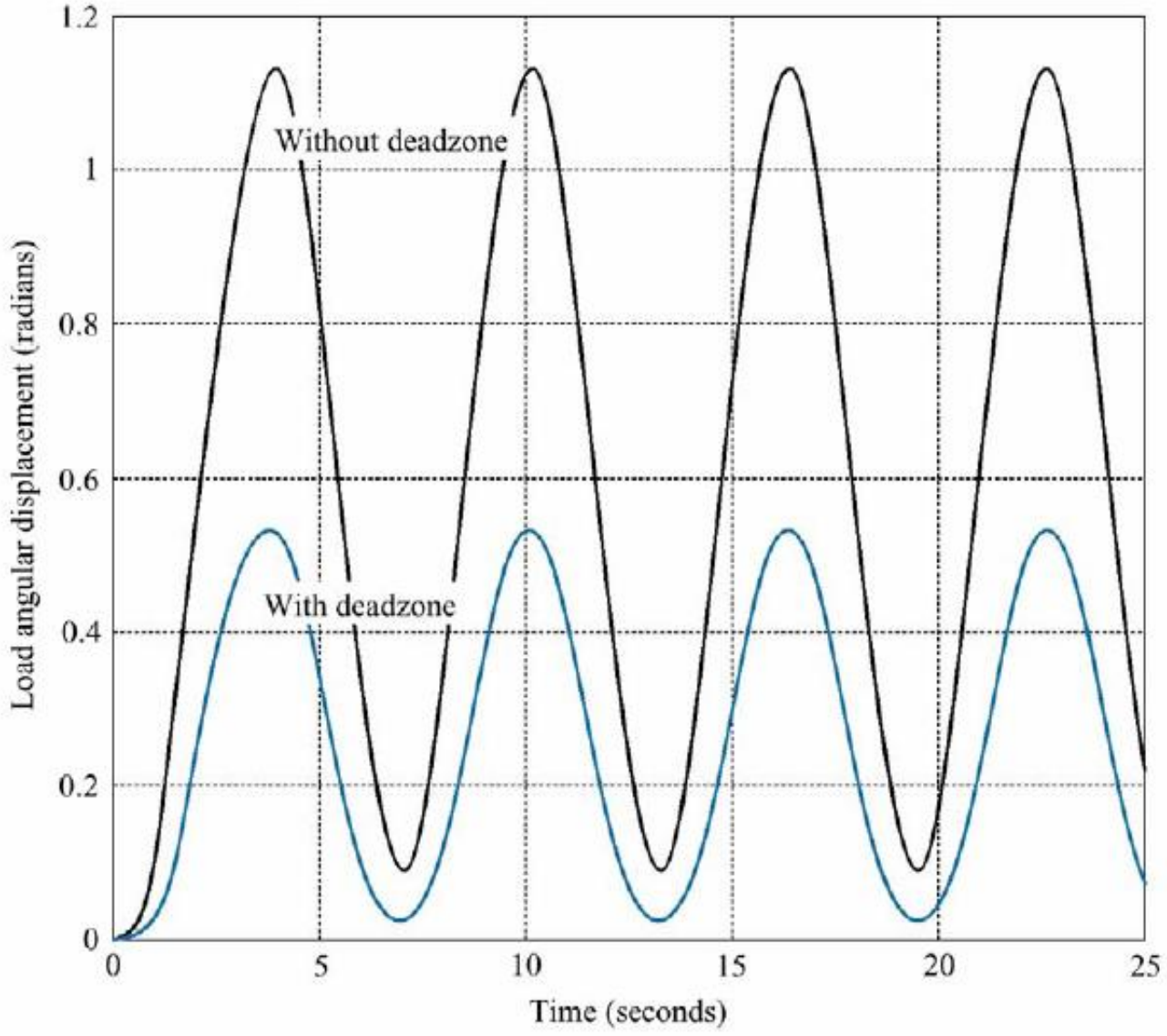
4.9 Effects of Nonlinearities upon Time Response

Effect of Saturation



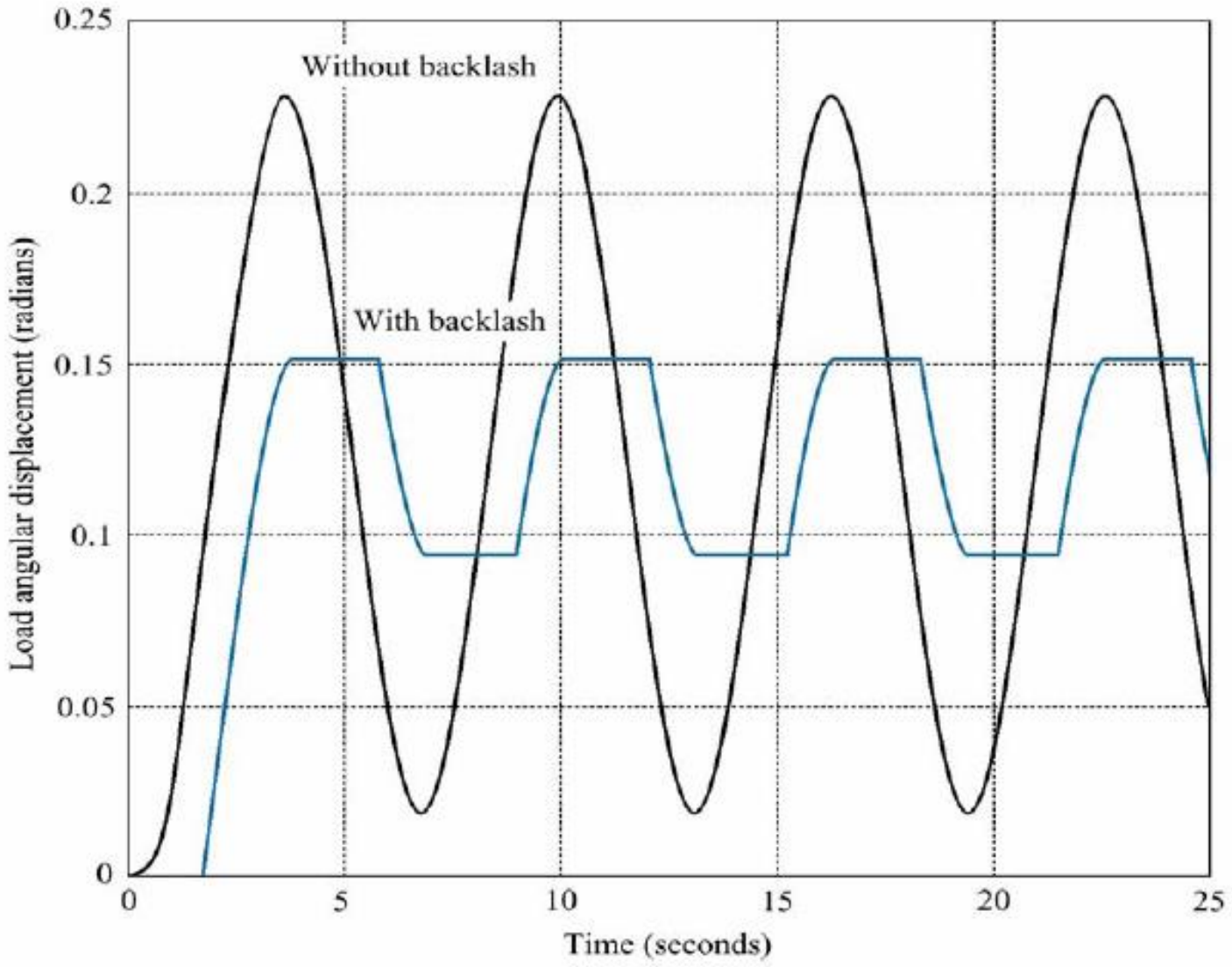
Effects of Nonlinearities upon Time Response

Effect of dead-zone



Effects of Nonlinearities upon Time Response

Effect of backlash



4.10 Laplace Transform Solution of State Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{x}(s) + \mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\,X(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = \frac{adj(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}\left[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)\right]$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Example 4.11

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 24 & 26 & s+9 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\begin{bmatrix} (s^2 + 9s + 26) & (s+9) & 1 \\ -24 & (s^2 + 9s) & s \\ -24s & -(26s + 24) & s^2 \end{bmatrix}}{s^3 + 9s^2 + 26s + 24}$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)\right]$$

$$X_1(s) = \frac{\left(s^3 + 10s^2 + 37s + 29\right)}{(s+1)(s+2)(s+3)(s+4)}$$

$$X_2(s) = \frac{\left(2s^2 - 21s - 24\right)}{(s+1)(s+2)(s+3)(s+4)}$$

$$X_3(s) = \frac{s\left(2s^2 - 21s - 24\right)}{(s+1)(s+2)(s+3)(s+4)}$$

$$Y(s) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = X_1(s) + X_2(s)$$

$$Y(s) = \frac{-6.5}{s+2} + \frac{19}{s+3} - \frac{11.5}{s+4}$$

$$y(t) = -6.5e^{-2t} + 19e^{-3t} - 11.5e^{-4t}$$

Poles : $-2,\ -3,\ -4$

4.11 Time Domain Solution of State Equations

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

$$\mathcal{L}\left[\mathbf{x}(t)\right] = \mathcal{L}\left[\mathbf{\Phi}(t)\mathbf{x}(0)\right] = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0)$$

$$\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \mathcal{L}^{-1}\left[\frac{\text{adj}(s\mathbf{I}-\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}\right] = \mathbf{\Phi}(t)$$

$\mathbf{\Phi}(t)$ is called State-Transition Matrix

Example 4.12

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \text{ with } \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since the characteristic equation is $s^2 + 6s + 8 = 0 \rightarrow s_1 = -2, s_2 = -4$

$$\boldsymbol{\Phi}(t) = \begin{bmatrix} \left(K_1 e^{-2t} + K_2 e^{-4t}\right) & \left(K_3 e^{-2t} + K_4 e^{-4t}\right) \\ \left(K_5 e^{-2t} + K_6 e^{-4t}\right) & \left(K_7 e^{-2t} + K_8 e^{-4t}\right) \end{bmatrix}$$

$$\boldsymbol{\Phi}(0) = \mathbf{I} \rightarrow K_1 + K_2 = 1$$

$$K_3 + K_4 = 0$$

$$K_5 + K_6 = 0$$

$$K_7 + K_8 = 1$$

$$\dot{\boldsymbol{\Phi}}(0) = \mathbf{A} \rightarrow -2K_1 - 4K_2 = 0$$

$$-2K_3 - 4K_4 = 1$$

$$-2K_5 - K_6 = -8$$

$$-2K_7 - 4K_8 = -6$$

$$\mathbf{\Phi}(t)=\begin{bmatrix} \left(2e^{-2t}-e^{-4t}\right) & \left(\frac{1}{2}e^{-2t}-\frac{1}{2}e^{-4t}\right) \\ \left(-4e^{-2t}+4e^{-4t}\right) & \left(-e^{-2t}+2e^{-4t}\right) \end{bmatrix}$$

$$\mathbf{\Phi}(t-\tau)\mathrm{B}=\begin{bmatrix} \left(\frac{1}{2}e^{-2(t-\tau)}-\frac{1}{2}e^{-4(t-\tau)}\right) \\ \left(-e^{-2(t-t)}+2e^{-4(t-t)}\right) \end{bmatrix}$$

$$\mathbf{\Phi}(t)x(0)=\begin{bmatrix} \left(2e^{-2t}-e^{-4t}\right) \\ \left(-4e^{-2t}+4e^{-4t}\right) \end{bmatrix}$$

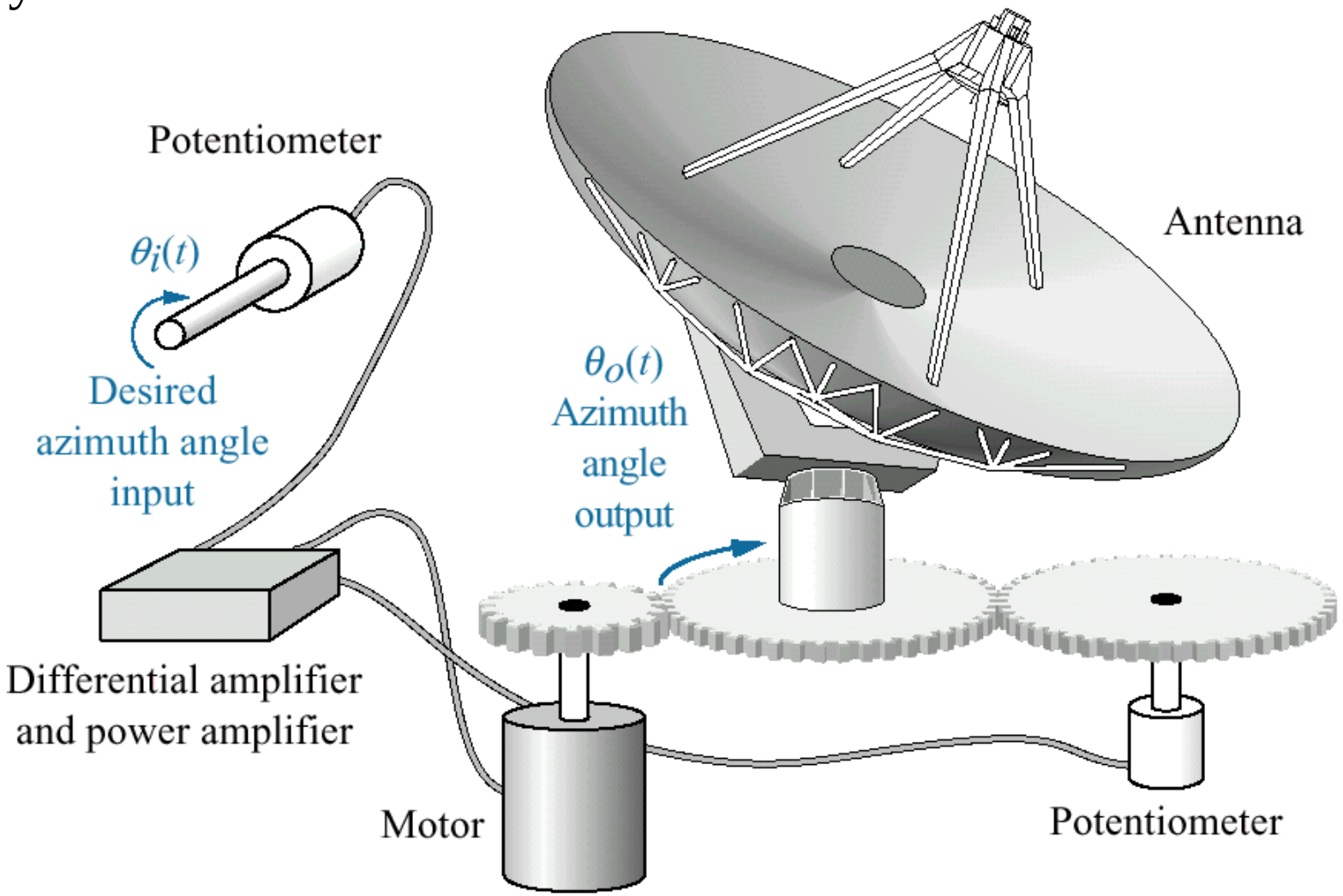
$$\int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau=\begin{bmatrix} \frac{1}{8}-\frac{1}{4}e^{-2t}+\frac{1}{8}e^{-4t} \\ \frac{1}{2}e^{-2t}-\frac{1}{2}e^{-4t} \end{bmatrix}$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

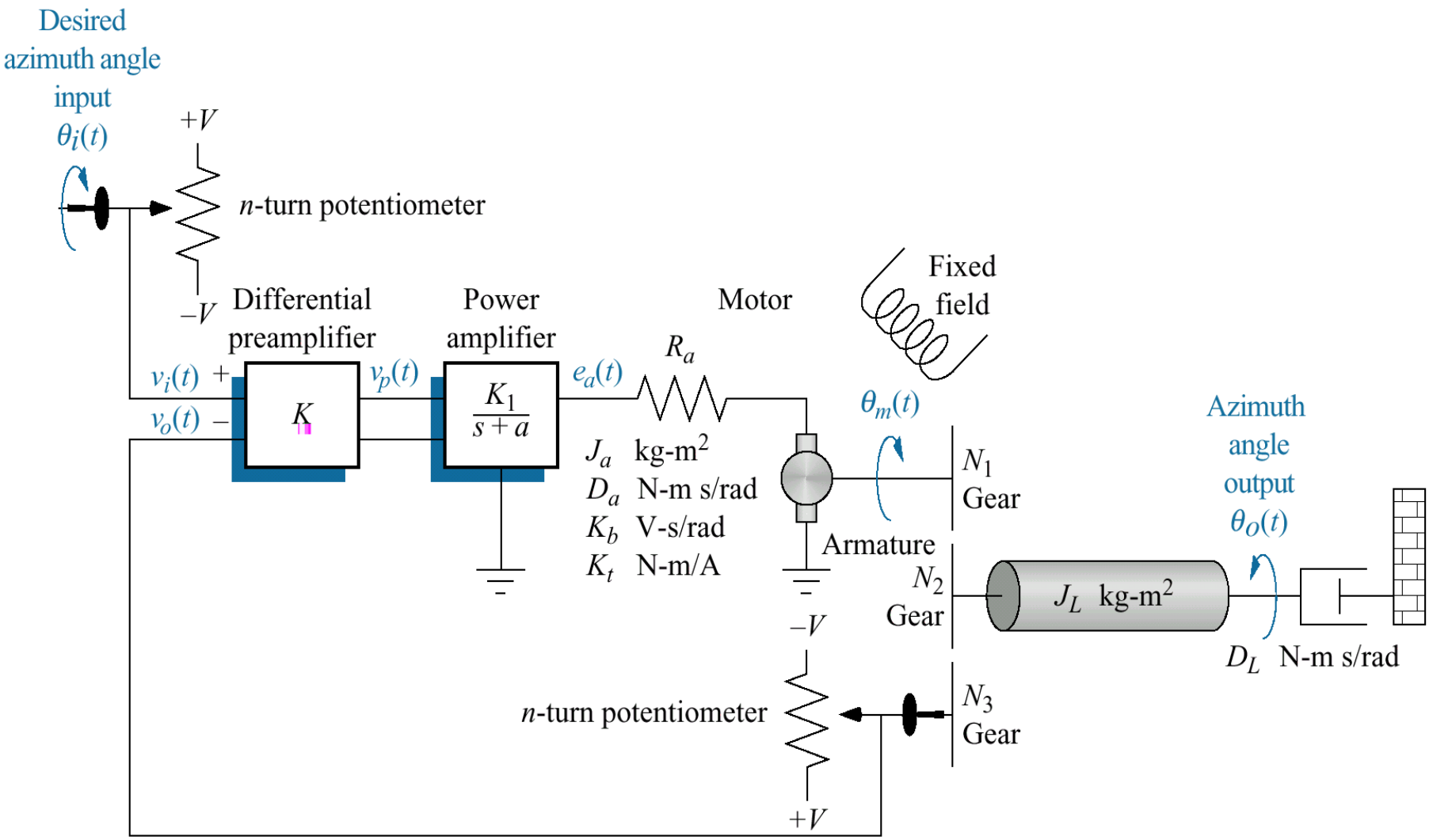
$$= \begin{bmatrix} \frac{1}{8} + \frac{7}{4}e^{-2t} - \frac{7}{8}e^{-4t} \\ -\frac{7}{2}e^{-2t} + \frac{7}{2}e^{-4t} \end{bmatrix}$$

Open-loop Response of Antenna Azimuth Position Control System

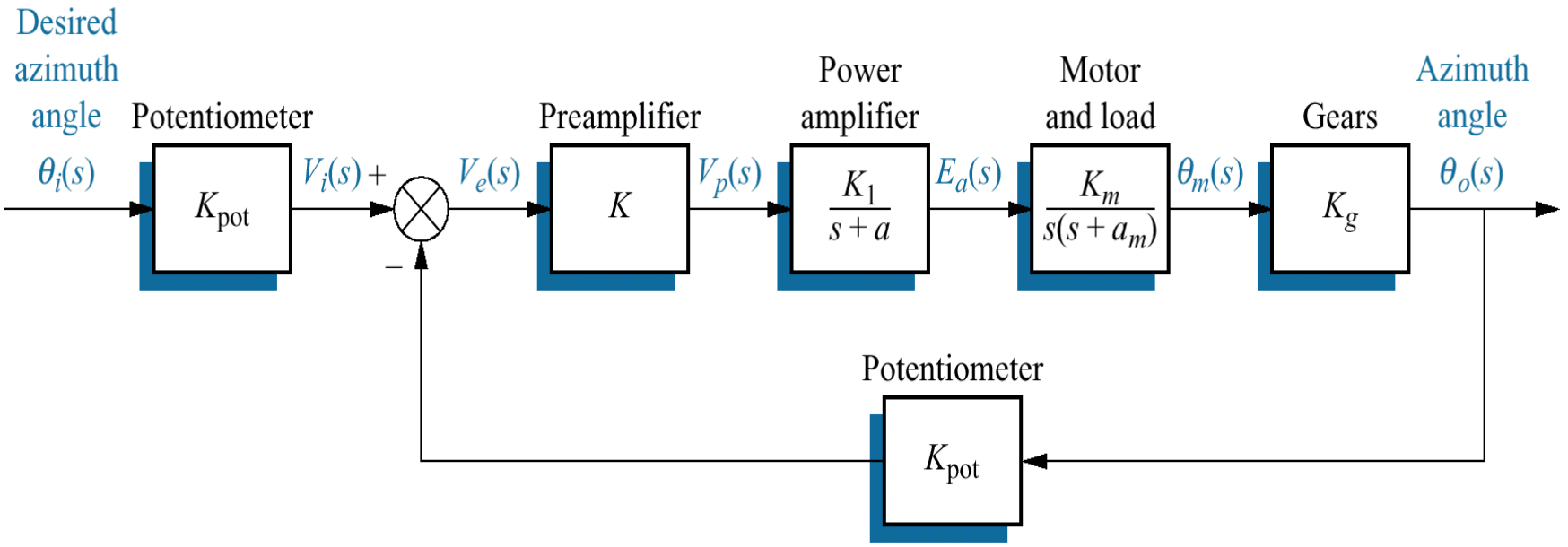
Layout

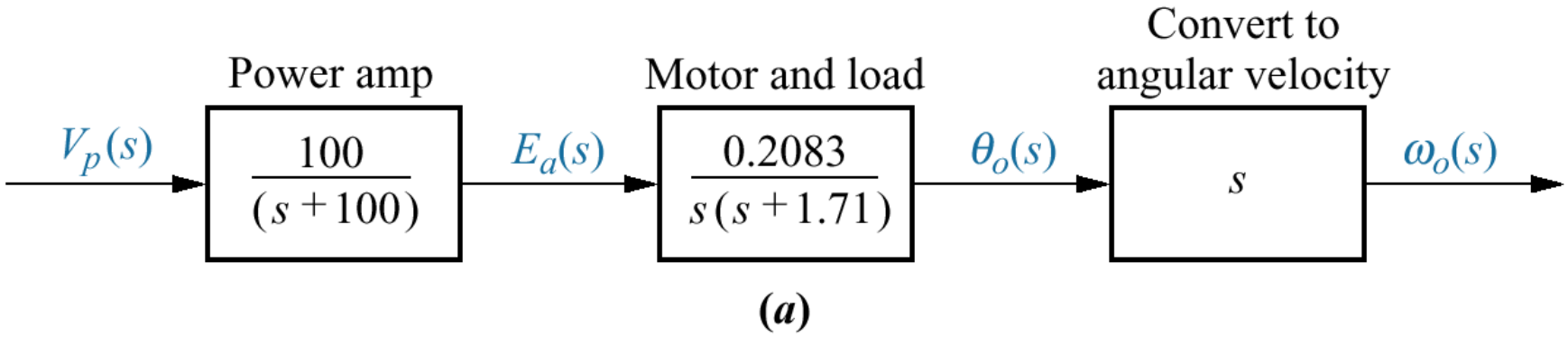


Schematic

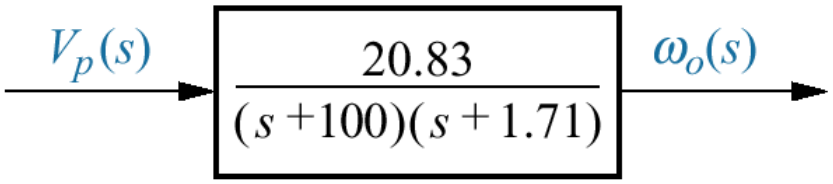

Desired azimuth angle input $\theta_i(t)$
$+V$
n-turn potentiometer
$-V$
Differential preamplifier
Power amplifier
Motor
Fixed field
$v_i(t)$ +
$v_o(t)$ –
K
$v_p(t)$
$\frac{K_1}{s+a}$
$e_a(t)$
R_a
J_a kg-m^2
D_a N-m s/rad
K_b V-s/rad
K_t N-m/A
$\theta_m(t)$
N_1 Gear
Armature
N_2 Gear
J_L kg-m^2
Azimuth angle output $\theta_O(t)$
D_L N-m s/rad
$-V$
n-turn potentiometer
N_3 Gear
$+V$

Block Diagram




Power amp
Motor and load
Convert to angular velocity
$V_p(s)$
$\frac{100}{(s+100)}$
$E_a(s)$
$\frac{0.2083}{s(s+1.71)}$
$\theta_o(s)$
s
$\omega_o(s)$

(a)


$V_p(s)$
$\frac{20.83}{(s+100)(s+1.71)}$
$\omega_o(s)$

(b)

Response of the system to step input

$$\omega_o = \underbrace{A}_{\text{Steady-state response}} + \underbrace{Be^{-100t} + Ce^{-1.17t}}_{\text{Transient Response}}$$

Note the poles are -100 and -1.17

In order to find the ξ and ω_n the denominatior of the transfer function must be expanded.

$$G(s) = \frac{20.83}{s^2 + 101.71s + 171}$$

$$\omega_n = \sqrt{171} = 13.08 \text{ and } 2\xi\omega_n = 101.71 \rightarrow \xi = 3.89 \ (\text{Overdamped})$$

Angular velocity response to step input

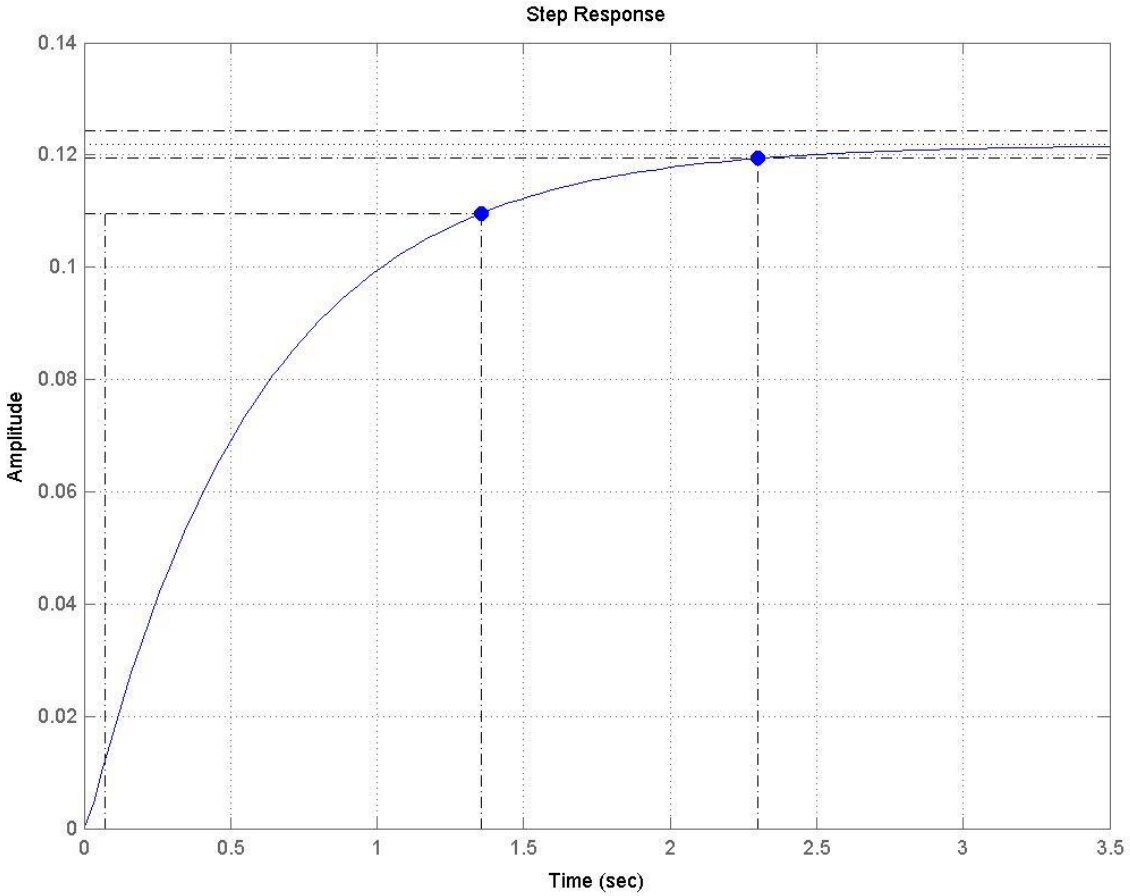
$$\omega_o(s) = \frac{20.83}{s\left(s^2 + 101.71s + 171\right)}$$

Partial fraction expanstion

$$\omega_o(s) = \frac{0.122}{s} + \frac{2.12\text{x}10^{-3}}{s+100} - \frac{0.124}{s+1.71}$$

Time domain

$$\omega_o(t) = 0.122 + \left(2.12\text{x}10^{-3}\right)e^{-100t} - 0.124e^{-1.71t}$$



Convert the transfer function into the state-space represntation

$$\frac{\omega_o(s)}{V_p(s)} = \frac{20.83}{s^2 + 101.71s + 171}$$

Inverse Laplace transform

$$\ddot{\omega}_o + 101.71\dot{\omega}_o + 171\omega_o = 20.83v_p$$

define the state varables as

$$x_1 = \omega_o$$

$$x_2 = \dot{\omega}_o$$

State equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -171x_1 - 101.71x_2 + 20.83v_p$$

where $v_p = 1$ (unit step input)

output equations $y = x_1$

Homework

- 4.6
- 4.10
- 4.19
- 4.25
- 4.50
- 4.60
- Matlab problem
  - » 4.7
  - » 4.11

4.2 Poles, Zeros, and System Response

Consider the step response of a first-order system.

$$C(s) = R(s)G(s) = \left(\frac{1}{s}\right)\left(\frac{s+2}{s+5}\right) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

Force response

Natural response